Credit Spread Bounds and Their Implications for Credit Risk Modeling[1]

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Abstract

Analogous to the yield curve which is essential in pricing interest rate contingent claims, the default probability curve is indispensable for pricing credit derivatives. However, when such a curve is generated by calibrating credit models to observed credit spreads, negative default probabilities often occur, which indicates a violation of no arbitrage. In this paper we present a formal treatment of this negative probability problem known in the credit researcher community. Specifically, for a large class of credit models we derive a set of analytical bounds within which a model will be consistent with an observed term structure of credit spreads (in the sense that the model-implied default probabilities are well defined). We also obtain some general results regarding the implications of the shape of the observed credit spread curve on the consistency of a credit risk model. In particular, we identify a class of models that are consistent only when the observed credit spread curves are flat at the long end.

1 Introduction

Credit derivatives constitute probably the largest derivatives market today. A recent survey by the Risk magazine (1999) estimates the market size of global credit derivatives to be $477 billion.[1] Credit derivatives make it possible to effectively package and transfer credit risk from one party to another. There are two major kinds of credit derivative contracts: default-triggered and spread-triggered. Default-triggered credit derivatives, e.g. default swaps and default baskets, incur cash flows when default (or a less severe credit event) occurs. Spread-triggered credit derivatives, e.g. spread options, incur cash flows when the underlying spread crosses a pre-specified level.

The first and perhaps also the most important task in pricing credit derivatives is to build the default probability curve. Such a curve is a collection of default probabilities over different horizons, analogous to the yield curve being a collection of yields of different terms. For a given issuer, its default probability curve (if available) describes how likely the issuer should default for any point in time in the future. Given the default probability curve, the valuation of default-triggered contracts is straightforward. Since they incur cash flows upon default only, the valuation involves simply calculating expected payoffs of potential payments. For spread-triggered contracts, which often have options features, the valuation involves both the default probability curve and the estimation of volatility parameters.

By market convention, default probabilities are computed off credit spreads. Normally, spreads on risky par floating-rate bonds (defaultable par floaters) are used to construct default probability curve.[2] We follow this approach in our analysis. (However, the analysis based on credit spreads from fixed-rate bonds is included in the appendix for completeness.) As

[1] See, for example, Caouette, Altman, and Narayanan (1998), Das (1995; 1999) and references therein on latest developments on credit derivatives.

[2] Due to the pricing bias in premium and discount bonds, it is preferable to obtain the default probability from spread curves of par bonds. Among par bonds, floating-rate bonds are more preferable than fixed-rate ones because floating-rate bonds are less sensitive to interest rate movements and, as a result, will be more likely to be near par throughout the life of the bond than fixed-rate bonds. Note that the risk-free floaters are always priced at par at coupon reset dates (provided that the coupon frequency coincides with the reset frequency). If a floater is priced at par and risky (defaultable), then its spread - difference between its floating coupons and the floating risk-free rate - should naturally represent the default risk and provide a clean measure of this risk.

There is a growing trend to use default swap spreads to construct the probability curve. In our case, default swap spreads are the same as floater spreads. Spreads of par asset swaps can also serve as the underlying instrument for computing default probabilities. Under certain conditions, investors can arbitrage among par floaters, par asset swaps, and default swaps if their spreads are not equal. As a result, practitioners generally do not differentiate among these spreads. See, for example, Chen and Soprozatti (2002) for a more detailed discussion of these three spreads.

mentioned earlier, using credit spreads to back out default/survival probabilities is similar to using yields to obtain discount factors. However, spread curves can be in many more varieties than yield curves can. For example, humped spread curves are not uncommon because it is typical for a company to survive long and well once it passes its critical period.[3] Also, spread curves with steep slopes are often observed. Sometimes, long term spreads can be as high as 10 times short term spreads. The spread level itself can be also very high. Some companies can maintain very high spreads without default.[4] Due to these characteristics of credit spread curves, constructing a default probability curve is a non-trivial task. In particular, it is known that negative default or survival probabilities frequently occur in the construction of the default probability curve. In practice, this negative probability problem is largely ignored and, to the best of our knowledge, a curve containing negative probabilities is often used regardlessly.

This paper provides a first attempt to investigate the issue of negative default or survival probabilities implied from credit spread curves. One objective of the paper is to provide a formal analysis of the issue. This would help us understand the nature of the negative probability problem and, as a result, see if this problem can be solved or at least alleviated. Another objective is to explore the implications of this analysis for credit risk modeling.

Specifically, we take as given an observed term structure of credit spreads on a particular issuer. We then consider a general class of credit risk models to determine under what conditions the *implied* default/survival probabilities from the observed spreads can become negative. We should emphasize that these models *a priori* do not allow negative default/survival probabilities.[5] When forced to fit the observed credit spread curve (i.e. when used to construct the default probability curve), a credit risk model can potentially have negative *implied* default/survival probabilities. If this does occur, the model is considered to be misspecified or inconsistent (with the observed credit spreads). The class of models we focus on in this study include many existing parametric models of defaultable bond prices as special cases. Specifically, the models to be analyzed here are those that can be characterized as having exogenously specified recovery rates as well as exogenously determined covariances between default risk and interest rates. Default risk in the models can be treated using either the reduced-form (Duffie

[3]This default pattern is also very common in mortgage defaults. For instance, the Federal Housing Administration experience often demonstrates a humped default curve.

[4]For example, during the 1998 Asian crises, Korean Development Bank bonds maintained spreads around 16% while the US treasury was not even 6%.

[5]Some reduced-form models of credit risk allow for negative hazard rates for analytical tractability. We do not consider such models in this analysis.

and Singleton (1999) and Jarrow and Turnbull (1995)) or the structural approach (Black and Scholes (1973) and Merton (1974)).

We establish a necessary and sufficient condition under which a particular model specification in the class of credit risk models studied here has negative implied default/survival probabilities. More specifically, we find that (1) Given an observed credit spread curve, each credit risk model specification is associated with two credit spread boundaries – which can be calculated explicitly; (2) A model admits no negative probabilities if and only if the observed credit spread curve lies between the model's two credit spread boundaries.

We then use the analytical formulas of the credit spread bounds to explore the implications of the shape of the observed credit spread curve for credit risk modeling. We show that (1) The class of credit risk models examined in our study are consistent only when observed spread curves for floating-rate bonds are flat at the long end; (2) A credit risk model will be inconsistent with upward-sloping (downward-sloping) spread curves if covariances between default risk and the reference interest rate are below (above) a certain threshold; (3) A model that restricts covariances to stay between the two thresholds can deal with spread curves that have a variety of shapes. The last property is a rather appealing attribute of the class of models analyzed here, given the recent empirical evidence showing a weak relationship between credit spreads and interest rates (see, for example, Duffee (1998, 1999)).

We also demonstrate how to implement our approach using some numerical examples. Finally, we apply our analysis to two actual credit spread curves (one from Santander BanCorp with a hump-shaped credit spread curve and the one from Softbank with a steeply upward-sloping credit spread curve).

The modeling approach used in this study builds on a simple default or no-default binomial process and covers many existing parametric models of defaultable bond prices. These include structural models such as Black and Cox (1976), Collin-Dufresne and Goldstein (2000), and Longstaff and Schwartz (1995) and reduced-form models such as Duffie and Liu (2001), Duffie and Singleton (1999), Jarrow and Turnbull (1995), Litterman and Iben (1991), and Madan and Unal (1998). Another example is Fons (1987), who directly assumes a particular functional form of the risk-neutral default probability. The "model-free" approach followed here can complement the existing parametric approaches. For instance, parametric models often assume a structure for covariances between default risk and the default-free interest rate. The credit spread bounds proposed here may provide some guidance on how to model these covariances.

The organization of the paper is as follows: In Section 2 We take as given observed credit spreads on defaultable par floating-rate bonds and derive credit spread bounds beyond which a credit risk model can generate negative *implied* probabilities. We also examine the implications of the bounds for credit risk modeling. Section 3 presents numerical and calibration results. Section 4 concludes. The analysis based on the observed spreads on fixed-rate bonds is included in the appendix. All proofs are also given in the appendix.

2 Defaultable Floating-Rate Bonds

The analysis in this section is based on the assumption that we observe a term structure of credit spreads on defaultable par floating-rate bonds. As mentioned before, we focus on floating-rate bonds because they will be more likely to be near par throughout the life of the bond than fixed-rate bonds.

2.1 Basic Setup

Let $\overline{T} > 0$ and $(\Omega, (\mathcal{F})_{t\in[0,\overline{T}]}, \mathbb{P})$ be a filtered probability space satisfying the *usual* conditions. Assume that there exist a set of default-free zero-coupon bond prices $P(t,T)$, $T < \overline{T}$ with $P(T,T) = 1\ \forall T$. Assume also there exists a default-free short rate process r such that the continuously compounding rate at time t of a money market account is $r(t)$.

Also given is a finite set of dates: $T_0 < T_1 < \cdots < T_{N-1} < T_N\ (< \overline{T})$. For simplicity and without loss of generality, assume that $T_0 = 0$ and the dates are equally spaced, i.e. $T_i - T_{i-1} = h\ \forall i \in I$, where the index set $I := \{1, \ldots, N\}$. Let $\ell(T_i, T_j)$, $0 \le i < j \le N$, represent the (annualized) default-free interest rate (e.g. the 6-month T-bill rate) over $[T_i, T_j]$ prevailing at T_i. By definition, $\ell(T_i,T_j) = \frac{1}{T_j - T_i}\left(\frac{1}{P(T_i,T_j)} - 1\right)$.

Denote by $\mathbb{Q}$ the risk-neutral measure relative to the process r. (See Harrison and Kreps (1979) and Harrison and Pliska (1981) for the existence of the $\mathbb{Q}$-measure.) We have $P(t,T) = E_t^{\mathbb{Q}}[\Lambda(t,T)]$, where $\Lambda(t,s) := \exp\left[-\int_t^s r(u)du\right], 0 \le t \le s \le T$, and $E_t^{\mathbb{Q}}[\cdot]$ represents the expectation conditional on the information set at time-t under the risk-neutral measure. It is often convenient to work under the forward measure when dealing with stochastic interest rates. Given a fixed T_n $(< \overline{T})$, and the time-0 price $P(0,T_n)$, the T_n-forward measure, denoted by $\mathbb{P}_n$, is equivalent to $\mathbb{Q}$ and the corresponding Radon-Nikodym derivative is given by (see, for example, Geman, El Karoui, and Rochet (1995) and Jamshidian (1987))

$$\frac{d\mathbb{P}_n}{d\mathbb{Q}} = \frac{\Lambda(0,T_n)}{P(0,T_n)} \tag{1}$$

Denote by $E^{\mathbb{P}_n}[\cdot]$ the expectation at time-0 under the T_n-forward measure.

Let τ be the default time, an $\mathbb{R}_+$-valued random variable. The event $\{\tau > t\}$ is then the event of no default by time t and the event $\{\tau > \bar{T}\}$ is the event of no default. Denote by $Q^{\mathrm{RN}}(0,T_n)$ and $Q^{P_n}(0,T_n)$ the unconditional probability of surviving by T_n under the risk-neutral measure and the T_n-forward measure, respectively. Then

$$Q^{\mathrm{RN}}(0,T_n) \quad = \quad E^{\mathbb{Q}}[I_{\{T_n<\tau\}}] \tag{2}$$

$$Q^{P_n}(0,T_n) \quad = \quad E^{\mathbb{P}_n}[I_{\{T_n<\tau\}}] \;=\; E^{\mathbb{Q}}\left[\frac{\Lambda(0,T_n)}{P(0,T_n)} I_{\{T_n<\tau\}}\right] \tag{3}$$

By definition, $Q^{\mathrm{RN}}(0,\cdot)$ and $Q^{P_n}(0,\cdot)$ should be in $[0,1]$. Probabilities defined in (2) and (3) can be calculated directly using a parametric model such as a reduced-form or a structural model or backed out from observed credit spreads. We follow mainly the latter approach in our analysis but present an example using the intensity-based approach in the appendix.

2.2 Implied Default and Survival Probabilities

Consider an n-period defaultable floating-rate bond with unit face value. The coupon rate is equal to a reference rate plus a credit spread of s_n, where $s_n \geq 0\,\forall n$. Assume that $\ell(.,.)$ is the reference interest rate of floating-rate bonds. Suppose also that the coupon payment dates and coupon reset dates coincide. In any given T_i-period with $i \leq n$, the payoff of an n-period defaultable floating-rate bond in the event of no default is $(\ell(T_i,T_{i+1})+s_n)h$ for $i<n$ and $1+(\ell(T_i,T_{i+1})+s_n)h$ for $i=n$. In the event of default, the payoff of the floating-rate bond is assumed to be w, where $0 \leq w \leq 1$. For simplicity, below w is assumed to be zero. This assumption is often made when using floating-rate bonds to construct a default-probability curve. Extending the analysis to a non-zero recovery rate is straightforward. Non-zero recovery rates will be considered for fixed-rate bonds in Appendix A.

Denote by $V(0,T_n)$ the time-0 value of an n-period defaultable par floating-rate bond. It is straightforward to show that

$$\begin{aligned} V(0,T_n) &= \sum_{i=1}^{n-1} E^{\mathbb{Q}}\left[I_{\{T_i<\tau\}}(\ell(T_{i-1},T_i)h + s_n h)\Lambda(0,T_i)\right] \\ &\quad + E^{\mathbb{Q}}\left[I_{\{T_n<\tau\}}(1+\ell(T_{n-1},T_n)h + s_n h)\Lambda(0,T_n)\right] \end{aligned} \tag{4}$$

Under the forward measure, equation (4) can be rewritten as follows

$$\begin{aligned} V(0,T_n) &= \sum_{i=1}^{n} P(0,T_i)\left[Q^{P_i}(0,T_i)(F(0,T_{i-1},T_i)^{-1}-1) + h\,\gamma_{i-1,i}\right] \\ &\quad + s_n\, h \sum_{i=1}^{n} P(0,T_i)Q^{P_i}(0,T_i) + P(0,T_n)Q^{P_i}(0,T_n) \end{aligned} \tag{5}$$

where $F(0,T_{i-1},T_i)$, the time-0 forward price, is given by

$$F(0,T_{i-1},T_i) = \frac{P(0,T_i)}{P(0,T_{i-1})} \tag{6}$$

and

$$\gamma_{i-1,i} := cov^{\mathbb{P}_i}\left(I_{\{T_i<\tau\}}, \ell(T_{i-1}, T_i)\right), \qquad i = 1,\ldots,N. \tag{7}$$

The following result is also used in the derivation of (5) (c.f. Musiela and Rutkowski (1997))

$$E^{\mathbb{P}_i}[\ell(T_{i-1},T_i)h] = \frac{1}{F(0,T_{i-1},T_i)} - 1 \tag{8}$$

Since the n-period defaultable floating-rate bond is priced at par at time-0, we have

$$V(0,T_n) = 1 \tag{9}$$

Equations (5) and (9) together determine the credit spread s_n.

Often, the valuation formula (5) along with condition (9) is used to back out a term structure of (implied) default/survival probabilities from observed credit spreads. We shall follow this approach since our focus is on implied default/survival probabilities and their implications for credit risk modeling. Under this approach, we take as given a term structure of credit spreads $(s_n)_{n\in I}$ and the default-free term structure $(P(0,T_n))_{n\in I}$. A model of defaultable floating-rate bond prices is now characterized by a specification of covariances $\{\gamma_{n-1,n}; n \in I\}$ and the recovery rate w. A default/survival probability curve can be then constructed using (5).

Proposition 1 *Consider a set of credit spreads, $\{s_n; s_n \geq 0, n = 1, \ldots, N\}$, where s_n represents the level of spread on an n-period defaultable par floating-rate bond. Assume a zero recovery rate. Then the T_n-forward survival probability over $[0, T_n), n = 1, \ldots, N$, is given by*

$$Q^{P_n}(0, T_n) = Q^{P_{n-1}}(0, T_{n-1}) \, \frac{1 - h\,\gamma_{n-1,n} \frac{F(0,T_{n-1},T_n)}{Q^{P_{n-1}}(0,T_{n-1})} - h\,(s_n - s_{n-1})\Sigma_{n-1}}{1 + h\, s_n\, F(0, T_{n-1}, T_n)} \tag{10}$$

where

$$\Sigma_n := \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{Q^{P_i}(0,T_i)}{Q^{P_n}(0,T_n)} \frac{P(0,T_i)}{P(0,T_n)} & n \geq 1 \\ 0 & n = 0 \end{cases} \tag{11}$$

Equation (10) along with the initial condition that $Q^{P_i}(0, 0) = 1$ provides a recursive formula for calculating survival probabilities (under the forward measures) implied from the observed term structure of credit spreads. Once survival probabilities are known, it is straightforward to calculate default probabilities.

2.3 Credit Spread Bounds

Although the formula given in (10) is easy to implement, survival/default probabilities determined from (10) are not always well defined for a given term structure of credit spreads. For instance, consider a model specification of $\gamma_{1,2} = 0$. One can see from (10) with $n = 2$ that the survival probability $Q^{P_i}(0, T_2)$ becomes negative if $s_2 - s_1 > 1/h$ (i.e. if the slope is too steep) and the forward default probability over $[T_1, T_2)$, $1 - Q^{P_2}(0, T_2)/Q^{P_1}(0, T_1)$, becomes negative if $s_2 < s_1/(1 + F(0, T_1, T_2))$ (meaning a steep downward slope). This indicates that when used to fit the observed credit spreads, certain model specifications may be misspecified in the sense that they admit negative default/survival probabilities. This is an illustration of the negative default probability issue which has been known in the practitioner community. Below we first discuss bounds on survival probabilities and then use them to derive credit spread bounds which can be used to detect model misspecifications (or inconsistency).

Consider first the general model specification, $\{\gamma_{n-1,n}; n \in I\}$.[6] In this case, being a probability, $Q^{P_i}(0,\cdot)$ has its natural lower and upper bounds, namely zero and one, respectively.

Consider next a model specification that can make bounds of $Q^{P_i}(0,\cdot)$ to be tighter than its natural ones. Under such a specification, covariances between default risk and the normalized money-market discount factor, $\Lambda(\cdot)/P(\cdot)$, are assumed to be the same across different horizons. (This model will be referred to as a constant-covariance model.) Formally, the model is based on the following assumption.

Assumption 1 *Let*

$$\rho_n := -cov^{\mathbb{Q}}\left(\frac{\Lambda(0,T_n)}{P(0,T_n)}, I_{\{T_n<\tau\}}\right), \qquad n = 1,\ldots,N \tag{12}$$

Then $\forall n \in I$, $\rho_n = \rho_c$ *and* ρ_c *is constant.*

Notice that one special case of the constant-covariance model is when $\rho_c = 0$, i.e., default risk and interest rates are uncorrelated. This, in fact, partially motivates Assumption 1.

Lemma 1 *Under Assumption 1, we have*

$$Q^{P_n}(0,T_n) \leq Q^{P_{n-1}}(0,T_{n-1}), \qquad n = 2,\ldots,N \tag{13}$$

Lemma 1 establishes that in the constant-covariance model, $Q^{P_n}(0,T_n)$ is also bounded above by $Q^{P_{n-1}}(0,T_{n-1})$, $n = 2,\ldots,N$. The intuition behind this result can be seen in the case of $\rho_c = 0$ where $Q^{P_n}(0,T_n) = Q^{\mathrm{RN}}(0,T_n)$ and, by definition, $Q^{\mathrm{RN}}(0,T_n)$ decreases in n. In general, equation (13) may not hold since $Q^{\cdot}(0,\cdot)$ are defined under different forward measures. One useful feature of the tighter upper bounds given by Lemma 1 is that they do not involve parameters ρ_n's explicitly. Once the bounds of $Q^{\cdot}(0,\cdot)$ are known, it is straightforward to derive credit spread bounds.

Theorem 1 *Let* $\{s_n; s_n \geq 0, n = 1,\ldots,N\}$ *be a set of credit spreads where each* s_n *denotes the spread of an n-period defaultable par floating-rate bond. Assume that the recovery rate* w *is zero. Let* $s_0 \equiv 0$.

[6] The recovery rate - part of a model specification - is not mentioned here since it is assumed to be zero throughout this section.

(i) The model specification, $\{\gamma_{n-1,n}; n \in I\}$, will be inconsistent with the observed credit spreads, $\{s_n; n \in I\}$, if $\exists n \in I$ such that s_n is not in $[\underline{s}_n^*, \overline{s}_n^*]$, where $\overline{s}_n^* \geq \underline{s}_n^*$ and

$$\underline{s}_n^* = s_{n-1} - \frac{1 - Q^{P_{n-1}}(0, T_{n-1}) + h\, F(0, T_{n-1}, T_n)\,(s_{n-1} + \gamma_{n-1,n})}{h\,[Q^{P_{n-1}}(0, T_{n-1})\Sigma_{n-1} + F(0, T_{n-1}, T_n)]} \tag{14}$$

$$\overline{s}_n^* = s_{n-1} + \frac{1}{h\,\Sigma_{n-1}} - \gamma_{n-1,n}\,\frac{1}{\Sigma_{n-1}}\frac{F(0, T_{n-1}, T_n)}{Q^{P_{n-1}}(0, T_{n-1})} \tag{15}$$

(ii) The constant-covariance model, $\{\gamma_{n-1,n}; \rho_n = \rho_c, n \in I\}$, will be inconsistent with the observed credit spreads, $\{s_n; n \in I\}$, if $\exists n \in I$ such that s_n is not in $[\underline{s}_n, \overline{s}_n]$, where $\overline{s}_n \geq \underline{s}_n$ and

$$\underline{s}_n = s_{n-1} - \frac{F(0, T_{n-1}, T_n)}{\Sigma_{n-1} + F(0, T_{n-1}, T_n)}\left(s_{n-1} + \frac{\gamma_{n-1,n}}{Q^{P_i}(0, T_{n-1})}\right) \tag{16}$$

$$\overline{s}_n = \overline{s}_n^* \tag{17}$$

As can be seen from Theorem 1, given $\{s_n; n \in I\}$, each model specification $\{\gamma_{n-1,n}; n \in I\}$ is associated with two boundaries, $\underline{s}^* := \{\underline{s}_n^*; n \in I\}$ and $\overline{s}^* := \{\overline{s}_n^*; n \in I\}$. Similarly, the constant-covariance model is also associated with boundaries, $\underline{s} := \{\underline{s}_n; n \in I\}$ and $\overline{s} := \{\overline{s}_n; n \in I\}$. Theorem 1 establishes that a model specification is misspecified if either of its boundaries is crossed by the observed credit spread curve. As a result, Theorem 1 provides a simple method to check the consistency of a model specification with the observed credit spreads.

2.4 Implications for Model Specifications

In this subsection we examine the implications of Theorem 1 for credit risk modeling. We show that given an observed credit spread curve, its slope can be used to pick a certain credit risk model that is free of negative probabilities. We shall consider four (classes of) models: (a) The general model: $\{\gamma_{n-1,n}; n \in I\}$; (b) The zero-covariance model: $\{\gamma_{n-1,n}; \rho_n = 0, n \in I\}$; (c) The constant-covariance model: $\{\gamma_{n-1,n}; \rho_n = \rho_c, n \in I\}$; (d) The constant-gamma model: $\{\gamma_{n-1,n}; \rho_n = \rho_c, \gamma_{n-1,n} = \gamma_0, n \in I\}$.

2.4.1 The General Model

One obvious observation is that among the four models mentioned above, the general model is the least restrictive and hence the most likely to be consistent with the observed credit spreads. This can be also seen from the fact that the "admissible band," $\{(\underline{s}_n^*, \overline{s}_n^*); n \in I\}$, is wider than the band, $\{(\underline{s}_n, \overline{s}_n); n \in I\}$. (It is easy to verify that $\underline{s}_n \geq \underline{s}_n^*$ $\forall n \in I$.)

Another observation from Theorem 1 is that if covariances between default risk and the reference interest rate of floating-rate bonds are "too negative," then a model will be misspecified. The reason is that $\forall n \in I$, conditions that $\overline{s}_n^* \geq \underline{s}_n^*$ impose an upper bound on $\gamma_{n-1,n}$. One can verify that the upper bound is given by

$$\overline{\gamma}_{n-1,n} = \frac{Q^{P_{n-1}}(0, T_{n-1})}{F(0, T_{n-1}, T_n)} \left[s_{n-1}\Sigma_{n-1} + \frac{1}{h}\left(1 + \frac{\Sigma_{n-1}}{F(0, T_{n-1}, T_n)}\right)\right] > 0, \qquad n = 1, \ldots, N. \quad (18)$$

As a result, a model misspecification will occur if $\exists n \in I$ s.t. $\gamma_{n-1,n} > \overline{\gamma}_{n-1,n}$. Note that there exists actually no cap on $\gamma_{0,1}$ because $\overline{s}_1^* = \infty$ but $\overline{\gamma}_{0,1}$ defined in (18) is not binding since $\gamma_{0,1} = 0$. Unless otherwise specified, assume in the balance of this section that $\gamma_{n-1,n} \leq \overline{\gamma}_{n-1,n}$ $\forall n \in I$.

One may also observe that the credit spread bounds obtained in Theorem 1, in fact, impose restrictions on the slopes of credit spread curves which the general model can generate. This observation allows us to analyze the implications of a credit spread curve that has a particular *shape* for credit risk modeling.

Corollary 1 *Consider the model specification, $\{\gamma_{n-1,n}; n \in I\}$. (The recovery rate w is zero.)*

(i) The model will be inconsistent with an upward-sloping credit spread curve, $\{s_n; s_{n-1} \leq s_n, n \in I\}$, if $\exists n \in I$ s.t. $\gamma_{n-1,n} > \gamma_{n-1,n}^u$, where

$$\gamma_{n-1,n}^u := \frac{1}{h} \frac{Q^{P_{n-1}}(0, T_{n-1})}{F(0, T_{n-1}, T_n)} \quad (19)$$

(ii) The model will be inconsistent with a downward-sloping spread curve, $\{s_n; s_{n-1} \geq s_n, n = 2, \ldots, N\}$, if $\exists n \in I$ s.t. $\gamma_{n-1,n} < \gamma_{n-1,n}^d$, where

$$\gamma_{n-1,n}^d := -s_{n-1} + \frac{1}{h} \frac{Q^{P_{n-1}}(0, T_{n-1}) - 1}{F(0, T_{n-1}, T_n)} \quad (20)$$

(iii) The model will be inconsistent with a flat spread curve, $\{s_n; s_1 = s_2 = \cdots = s_N \geq 0\}$*, if* $\exists n \in I$ *s.t.* $\gamma_{n-1,n} < \gamma^d_{n-1,n}$ *or* $\gamma_{n-1,n} > \gamma^u_{n-1,n}$.

Notice that $\gamma^d_{n-1,n} < 0 < \gamma^u_{n-1,n} < \overline{\gamma}_{n-1,n}, \forall n \in \{2, \ldots, N\}$. One implication of Corollary 1 is that the model specification, $\{\gamma_{n-1,n}; \gamma_{n-1,n} \in \left[\gamma^d_{n-1,n}, \gamma^u_{n-1,n}\right] \forall n \in I\}$, can be consistent with credit spread curves that are flat, upward-sloping, downward-sloping or hump-shaped, provided that these curves stay within their boundaries specified in Theorem 1. In particular, it is easy to see that the model, $\{\gamma_{n-1,n}; \gamma_{n-1,n} = 0, \forall n \in I\}$, is always consistent with flat credit spread curves.

Finally, let's analyze the implications of credit spread curves' behavior at the long end for credit risk modeling. The analysis is based on the corollary that follows.

Corollary 2 *Consider Theorem 1. Suppose N is large.* $\forall n \in I$*, if n is large, then* $\underline{s}^*_n \approx s_{n-1}$*,* $\overline{s}^*_n \approx s_{n-1}$*, and* $\underline{s}_n = \underline{s}^*_n \approx s_{n-1}$*.*

Corollary 2 implies that when n gets large, both the upper bound $\overline{s}^*_n$ and the lower bound $\underline{s}^*_n$ move towards s_{n-1}. As a result, for large n's, if $s_n \in [\underline{s}^*_n, \overline{s}^*_n]$ then s_n must be "very close" to s_{n-1}. Namely, the general model, $\{\gamma_{n-1,n}; n \in I\}$, can only be consistent with credit spread curves that flatten out. Put in another way, the class of models considered in this section cannot generate a sensible default probability curve from a credit spread curve that does not flatten out.

2.4.2 The Zero-Covariance Model

One specification widely used in the literature is that default risk is uncorrelated with interest rates,[7] i.e., $\rho_c = 0$ in our notation. Denote by $\underline{s}^0_n$ and $\overline{s}^0_n$ respectively the lower and upper bounds of the n-period spread s_n using the zero-covariance model. It follows from (16) and (17) that $\forall n \in I$,

$$\underline{s}^0_n = \frac{s_{n-1}\, \Sigma^0_{n-1}}{\Sigma^0_{n-1} + F(0, T_{n-1}, T_n)} \tag{21}$$

$$\overline{s}^0_n = s_{n-1} + \frac{1}{h\, \Sigma^0_{n-1}} \tag{22}$$

[7] e.g. Duffie and Singleton (1999) and Jarrow and Turnbull (1995).

where $\Sigma_n^0 = \Sigma_n|_{\gamma_{i-1,i}=0\ \forall i \le n}, \forall n \in I$. Note that, in general, $\Sigma_n^0 \ne \Sigma_n|_{\gamma_{n-1,n}=0}$, $\underline{s}_n^0 \ne \underline{s}_n|_{\gamma_{n-1,n}=0}$, and $\overline{s}_n^0 \ne \overline{s}_n|_{\gamma_{n-1,n}=0}$.

Now that $\gamma_{n-1,n} = 0\, \forall n \in I$, it follows from Corollary 1 that the zero-covariance model could be consistent with the observed credit spread curves that have a variety of shapes and, in particular, the model is always consistent with flat spread curves.

However, the credit spread bounds given in (21) and (22) do impose strong restrictions on the behavior of credit spread curves predicted by the zero-covariance model. For instance, since $s_{n-1} - \underline{s}_n^0$ is significantly smaller than $\overline{s}_n^0 - s_{n-1}$ when n is small (note that $\overline{s}_1^0 = \infty$), credit spread curves with a reasonable s_1 that stay within their boundaries will be much closer to the lower boundary than to the upper one at the front end (c.f. Figures 1 and 2). Also, as shown below, the zero-covariance model's boundaries, $\underline{s}^0 := \{\underline{s}_n^0; n \in I\}$ and $\overline{s}^0 := \{\overline{s}_n^0; n \in I\}$, flatten out much faster than other models' boundaries do. These results may explain the following observation known in the practitioner world: a credit risk model that assumes the recovery rate is zero and that interest rates and default risk are uncorrelated has difficulty in dealing with steeply sloped or humped credit spread curves.

2.4.3 The Constant-Covariance Model

In this model, $\gamma_{n-1,n}$'s do not have to be all zero. However, they also need to be capped because of conditions, $\overline{s}_n \ge \underline{s}_n\, \forall n \in I$. It is easy to verify that in the constant-covariance model, the cap of $\gamma_{n-1,n}\, \forall n \in I$ is also given by (18).

Like the general model analyzed earlier in Section 2.4.1, the constant-covariance model has implications on the shapes of credit spread curves it can generate. For instance, the model will be inconsistent with an upward-sloping credit spread curve if $\exists n \in I$ s.t. $\gamma_{n-1,n} > \gamma_{n-1,n}^u$; the model will be inconsistent with a downward-sloping credit spread curve if $\exists n \in I$ s.t. $\gamma_{n-1,n} < -s_{n-1} Q^{P_{n-1}}(0, T_{n-1})$ (since it follows from (16) that in this case $s_{n-1} < \underline{s}_n$). On a related note, when dealing with downward-sloping spread curves, the constant-covariance model is actually less appealing than the general model because, as mentioned earlier, $\underline{s}_n \ge \underline{s}_n^*\, \forall n \in I$.

One situation where the constant-covariance mode might be useful in particular is when the observed credit spreads indicate that the zero-covariance model is misspecified. In this situation it is possible to obtain a constant-covariance model that is consistent with the credit spread data. To do this, it will suffice to make sure the boundaries, $\underline{s}$ and $\overline{s}$, "sandwich" the observed credit spread curve. Below, we demonstrate how to achieve this by utilizing (16) and (17).

Suppose the observed credit spreads are such that $\exists n \in I$, $s_n < \underline{s}_n^0$, i.e., the zero-covariance model is misspecified. Consider the following specification of the constant-covariance model (specification S1): $\{\gamma_{n-1,n} > 0, \gamma_{i-1,i} = 0 \forall i < n\}$. As can be seen from (16) and (17), a positive $\gamma_{n-1,n}$ will lower both $\underline{s}_n$ and $\overline{s}_n$. If $\underline{s}_n^0$ and $\overline{s}_n^0$ are not close, then under S1, $s_n \in [\underline{s}_n, \overline{s}_n]$ (i.e. S1 is a consistent specification). Similarly, if a misspecification of the zero-covariance model is due to $s_n > \overline{s}_n^0$, then the specification (S2), $\{\gamma_{n-1,n} < 0, \gamma_{i-1,i} = 0 \forall i < n\}$, will raise $\overline{s}_n$ and may result in $s_n \in [\underline{s}_n, \overline{s}_n]$. Specifications S1 and S2 illustrate the idea of introducing "local" covariances near the location where $\underline{s}^0$ or $\overline{s}^0$ is crossed by the observed credit spread curve. Such specifications are convenient but *ad hoc* and may also be difficult to incorporate a dynamic model of the reference interest rate.

Another specification of the constant-covariance model that may improve the zero-covariance model is $\{\gamma_{n-1,n} = \gamma_0 \ \forall n \geq 2\}$, where γ_0 is constant. Denote this the constant-gamma model. Unlike S1 and S2, this specification may be consistent with a model of the reference interest rate, which itself can then be calibrated using the credit spread data. As indicated by the next proposition, the constant-gamma model is particularly suitable when the misspecification of the zero-covariance model is due to its upper spread boundary being crossed.

Proposition 2 *Consider $\{s_n; s_n \geq 0, n = 1, \ldots, N\}$ where each s_n denotes the credit spread of an n-period defaultable par floating-rate bond. Assume that the recovery rate is zero. Suppose $s_i < \overline{s}_i^0, \forall i \in \{2, \ldots, n-1\}, n = 3, \ldots, N$.*

(i) If $\gamma_{i-1,i} < 0 \ \forall\, i \in \{2, \ldots, n\}$, then $\overline{s}_i > \overline{s}_i^0 \, \forall i \in \{2, \ldots, n\}$, where $n = 2, \ldots, N$.

(ii) If $\gamma_{i-1,i} > 0 \ \forall\, i \in \{2, \ldots, n\}$, then $\overline{s}_i < \overline{s}_i^0 \, \forall i \in \{2, \ldots, n\}$, where $n = 2, \ldots, N$.

Part (i) of Proposition 2 establishes that if $\overline{s}^0$ lies above the observed credit spread curve until at least period-$(n-1)$, then $\overline{s}$ using the constant-gamma model with a negative γ_0 will lie above $\overline{s}^0$ until at least period-n. As a result, if the observed spread curve first crosses $\overline{s}^0$ in period-n (i.e., $s_n > \overline{s}_n^0$), the constant-gamma model with a negative γ_0 may turn out to be correctly specified. The intuition behind this is as follows. The fact that $s_n > \overline{s}_n^0$ indicates that the credit spread at period-n implied from the zero-covariance model is too low. By introducing a positive correlation between default risk and the reference interest rate, the constant-gamma model with negative γ_0 can potentially generate a high enough $\overline{s}_n$. (c.f. Figure 5 for an example on how this works.) Together, parts (i) and (ii) of Proposition 2 show that in the constant-gamma model, $\forall n \geq 2, \overline{s}_n - \overline{s}_n^0$ and γ_0 have opposite signs.

Unfortunately, the sign of $\underline{s}_n - \underline{s}_n^0$ for a given $n \in \{2, \ldots, N\}$ is, in general, not fixed. To see this, we have from (16) and (21)

$$\begin{aligned} \underline{s}_n - \underline{s}_n^0 &= s_{n-1} \frac{F(0, T_{n-1}, T_n)\,(\Sigma_{n-1} - \Sigma_{n-1}^0)}{(\Sigma_{n-1} + F(0, T_{n-1}, T_n))(\Sigma_{n-1}^0 + F(0, T_{n-1}, T_n))} \\ &\quad - \frac{\gamma_{n-1,n}}{\Sigma_{n-1} + F(0, T_{n-1}, T_n)} \frac{F(0, T_{n-1}, T_n)}{Q^{P_{n-1}}(0, T_{n-1})} \end{aligned} \tag{23}$$

Since $\Sigma_{n-1} - \Sigma_{n-1}^0$ and $\gamma_{n-1,n}$ have the same sign (see the proof of Proposition 2), the right-hand side (RHS) of (23) has an ambiguous sign. However, the sign of $\underline{s}_n - \underline{s}_n^0$ should be determined largely by the sign of $\gamma_{n-1,n}$. The reason is that on the RHS of (23), the last term normally dominates the first term since, in general, $\Sigma_{n-1} - \Sigma_{n-1}^0$ is small and Σ_{n-1}^0 is large. As a result, when the misspecification of the zero-covariance model is due to $\underline{s}^0$ being crossed from above, the constant-gamma model with positive γ_0 may well be consistent with the observed credit spreads. (c.f. Figure 4 on such an example.)

Finally, the constant-gamma model dominates the zero-covariance model at the long end of a given credit spread curve. As indicated by Corollary 2, both the zero-covariance and constant-gamma models have difficulty dealing with credit spread curves that are steeply sloped at the long end. However, the credit spread bounds from the former model flatten out much faster than the bounds from the latter model do. To see this, notice that both $(\underline{s}_n - s_{n-1})/(\underline{s}_n^0 - s_{n-1})$ and $(\overline{s}_n - s_{n-1})/(\overline{s}_n^0 - s_{n-1})$ are in the order of magnitude of $Q^{P_{n-1}}(0, T_{n-1})^{-1}$ when n is large (c.f. the proof of Corollary 2). Recall that $Q^{P_{n-1}}(0, T_{n-1})$ is small for a large n. As a result, the constant-gamma model is more appealing than the zero-covariance model in dealing with the long ends of credit spread curves.

3 Numerical Results

In this section we discuss first the implementation of our method using numerical examples. We then apply the method to two actual OTC credit spread curves.

Recall that in this analysis we take as given a term structure of credit spreads $(s_n)_{n\in I}$ and the default-free term structure $(P(0,T_n))_{n\in I}$, and select a credit risk model to build a default probability curve. Then the issue is whether this particular model admits negative (implied) probabilities in such applications. This can be analyzed using the credit spread bounds obtained in Section 2.2.

Specifically, given $(s_n)_{n\in I}$, $(P(0,T_n))_{n\in I}$, and a credit risk model $\{\gamma_n; n\in I\}$ (and recovery rate, which is assumed to be zero), equations (11), (10), (14), and (15) will be used to calculate the survival probability curve and its upper/lower bounds. For instance, in a given time period, say period-n, one calculates Σ_{n-1} first using (11), then $Q(0,T_n)$ next using (10), and then Σ_n using again (11). Doing this recursively for each period will yield the entire survival probability curve. The bounds can be obtained in a similar fashion. Since formulas (11), (10), (14), and (15) are explicit, the implementation can be done easily in a spreadsheet.

3.1 Numerical Examples

In order to see the intuition behind Theorem 1, we examine a variety of credit spread curves, their corresponding upper and lower bounds, and their implied term structure of default/survival probabilities. In particular, we consider spread curves of three commonly observed shapes: flat, upward-sloping, and steeply upward-sloping. As mentioned earlier, the shape and level of credit spread curves are much richer than those of default-free yield curves.

Figure 1 presents the plots of a flat credit spread curve and its lower and upper boundaries. The spread is set to be 50 basis points. Covariances between interest rates and default risk are assumed to be zero ($\rho_c = 0$). One can see from the plots that the credit spread curve lies between its lower and upper boundaries over the 30-year horizon. This implies that survival probabilities are well defined and the zero-covariance model is consistent with the credit spread curve. One can also see from the figure that both boundaries are very steep at the short end. For instance, as shown in Figure 1(a), the upper bound begins with more than 200% at one year and then drops to near 50% at the two-year point. On the other hand, as shown in Figure 1(b), the lower bound jumps from near 25 basis points at one year to 38 basis points at the two-year point. Notice that both bounds converge to the spread curve at the long end.

Figure 2 illustrates the plots of an upward-sloping credit spread curve and its lower and upper boundaries using the zero-covariance model. The spread curve is chosen to be upward-sloping from 2 basis points gradually to 80 basis points. One can see from the plots that in this case, the spread curve also lies between its lower and upper bounds. This indicates that the zero-covariance model can also be consistent with credit spread curves that have "mild" slopes. One can also see from the plots that the two bounds of an upward-sloping spread curve behave similarly to their counterparts for a flat spread curve except at the short end. For instance, both the lower and upper bounds of an upward-sloping spread curve converge to the spread curve itself at the long end. Also, as shown in Figure 2(a), the upper bound of an upward-sloping spread curve has a shape similar to that of the upper bound of a flat spread curve. However, as shown in Figure 2(b), the lower bound of an upward-sloping spread curve is fairly close to the spread curve at the short end.

Figure 3 illustrates the plots of a steeply upward-sloping credit spread curve, its upper bound, and its implied survival probability curve generated using the zero-covariance model. One can see from the figure that around year 24, the spread curve crosses its upper bound and, as a result, survival probabilities become negative. This indicates that the zero-covariance model is inconsistent with the given credit spread curve.

3.2 Two Actual Spread Curves

In this subsection we analyze two actual cases: one is Santander BanCorp (SBP) with a humped credit spread curve; the other is Softbank with a steeply upward-sloping spread curve. Both curves were obtained from the Structure Credit Trading of Lehman Brothers Inc.

We observe eleven spreads in both cases. These spreads match with Constant Maturity Treasury rates (CMT) that have the following eleven maturity dates: 3 months, 6 months, 1 year, 2 years, 3 years, 4 years, 5 years, 7 years, 10 years, 20 years, and 30 years. It should be noted that not all of the eleven spreads are liquidly traded. Some of them actually are merely trader marks and do not have actual contracts. Nevertheless, for our purposes, we assume that trader beliefs should represent fair market values as liquidity is low.

To proceed, we need a term structure of 60 semi-annual yield spreads. We employ a widely used linear-smoothing methodology in the industry (e.g. Bloomberg) to calculate those 60 spreads from the 11 constant maturity spreads obtained from Lehman. Namely, a linear interpolation is used to solve for the remaining 49 spreads such that the 11 coupon bonds, whose coupon rates equal those CMT rates and that mature on those CMT dates, are priced at par.

Consider first the case of SBP. The 11 credit spreads on SBP, measured against the U.S. CMT rates, were updated on July 26, 1999. The spreads and other data used to calibrate SBP's credit spread curve are reported in Table 1. Figure 4 illustrates the plots of SBP's credit spread curve, its lower bound, and its implied survival probability curve generated using the zero-covariance model. As shown in Figure 4(a), the spread curve is U-shaped that begins at 25 basis points at year one, is flat at zero basis points from year 7 to year 20, and then rises to 13 basis points in year 30. The spread curve crosses its lower bound from the zero-covariance model around the 3.5-year point, when survival probabilities become greater than one (or default probabilities become negative). This indicates that the zero-covariance model is inconsistent with the SBP credit spread curve. As discussed in Section 2.4.3, the constant-gamma model with positive γ_0 may be "correct" for the observed credit spreads here. Figure 4(b) shows the plots of the SBP credit spread curve, its lower bound, and its implied survival probability curve using the constant-gamma model with $\gamma_0 = 0.01$. One can see from the figure that this specification is indeed consistent with the observed credit spread curve.

Consider next the example of Softbank, whose spread curve (dated 6/8/00) is steeply upward-sloped. The data used in this example are also reported in Table 1. Figure 5 illustrates the Softbank credit spread curve, its upper bound, and its implied survival probability curve. The spread curve begins with near 48 basis points at year 1, increases to about 171 basis points at year 10, and then rises to over 400 basis points at year 30. (In order to increase the resolution near the long end, the portion of the spread curve before year 10 is not shown in the figure.) If one recalls Figure 3(a), it is not surprising to see from Figure 5(a) that at the long end the spread curve crosses its upper bound generated using the zero-covariance model and survival probabilities become negative. As shown in Figure 5(b), the constant-gamma model with $\gamma_0 = -0.005$ is free of negative survival probabilities and correctly specified.

To summarize, we have demonstrated in this section that our analysis provides a simple method to identify inconsistent models of defaultable bond yields and construct default-probability curves that admit no negative probabilities.

4 Conclusions

Construction of the default probability curve is the first and perhaps also the most fundamental step in pricing defaultable claims such as credit derivatives. In this paper, we examine a well-known and challenging issue faced by credit derivative researchers – how to build a default probability curve which is free of negative probabilities. Negative probabilities occur because credit spread curves that are used to back out *implied* default/survival probabilities are not uncommon to take extreme shapes. Steeply-sloped (humped) spread curves tend to generate negative survival (default) probabilities. The two actual credit spread curves (from SBP and Softbank) analyzed in the paper both can have negative implied probabilities.

We consider a large class of models of defaultable bond prices and for any given term structure of observed credit spreads, derive a set of analytical credit spread bounds beyond which a model will be inconsistent (in the sense that it will admit negative implied probabilities). The class of models considered in this study include many existing parametric models (both structural or reduced-form ones) as special cases. The term structure of observed credit spreads is assumed to come from either par floating-rate bonds or par fixed-rate bonds.

Using the analytical formulas for credit spread bounds, we are also able to obtain some general results regarding the implications of the shape of the observed credit spread curve on the consistency of a credit risk model. For instance, we identify a class of credit models that are consistent only when the observed credit spread curves for floating-rate bonds flatten out.

In conclusion, we present a formal analysis of the negative default probability problem in pricing credit derivatives and provide a simple approach for judging the consistency of credit models. Our analysis may also provide some guidance on how to improve existing parametric credit models.

A Defaultable Fixed-Rate Bonds

In this appendix we consider a set of observed risky yield spreads from fixed-rate par bonds. The basic setup is the same as described in Section 2.1. Unlike Section 2, however, non-zero recovery rates will be allowed here. In particular, we shall consider two alternative recovery specifications. One specification is based on the face value of the bond (see below). The other specification is based on the market value (Duffie and Singleton (1999)).

A.1 Models Based on Recovery of Face Value

Consider an n-period defaultable bond with unit face value. The bond pays fixed-rate coupons and matures at T_n. Let c_n be the coupon rate. In the event of default the bond recovery is equal to w (times the face value) and to be received on the first scheduled coupon date after default (discrete-time recovery in the sense of Duffie (1998)). For zero-coupon bonds, this assumption is equivalent to Jarrow and Turnbull's (1995) model of recovery of treasury.

A.1.1 Survival Probabilities

Let $V_B^{\mathrm{RF}}(0,T_n)$ denote the time-0 value of the bond with discrete-time recovery. We have

$$\begin{aligned} V_B^{\mathrm{RF}}(0,T_n) &= \sum_{i=1}^{n} c_n h\, P(0,T_i) Q^{P_i}(0,T_i) + P(0,T_n) Q^{P_n}(0,T_n) \\ &\quad + \sum_{i=1}^{n} w\, P(0,T_i) \left[Q^{P_{i-1}}(0,T_{i-1}) - Q^{P_i}(0,T_i) \right] \end{aligned} \tag{24}$$

where $Q^{P_i}(0,T_i)$, as before, represents the unconditional survival probability by time T_i under the T_i-forward measure, and h^{-1} is the coupon frequency. On the RHS of (24), the first two terms represent the payoff conditional on no default, whereas the last term comes from the bond recovery in the event of default.

Implied survival probabilities can be obtained from (24) plus the par bond conditions that $c_n = y_n$ and $V_B^{\mathrm{RF}}(0,T_n) = 1$. Following the proof of Proposition 1, we have

$$(1 + y_n h - w)\, \frac{Q^{P_n}(0,T_n)}{Q^{P_{n-1}}(0,T_{n-1})} = -w + \frac{1 + h\ (y_{n-1} - y_n)\Sigma_{n-1}}{F(0,T_{n-1},T_n)}, \qquad n = 1,\ldots,N, \tag{25}$$

where Σ_n is defined as earlier in (11). Given the initial condition that $Q(0,0) = 1$, equation (25) can be used to calculate $(Q^{P_n}(0,T_n))_{n\in I}$.

A.1.2 Risky Yield Bounds

As before, bounds of $Q(0,\cdot)$ depend on the restrictions imposed on covariance coefficients ρ_n's. Specifically, we shall consider the bounds under two models: the general recovery-of-face (RF) model that imposes no restrictions on ρ_n's and the constant-covariance RF model that imposes Assumption 1 on ρ_n's. Once the bounds of $Q(0,\cdot)$ are determined, they can be translated to credit spread bounds using (25). However, unlike in Section 2, here it is more convenient to do this indirectly by deriving risky yield bounds first.

Theorem 2 *Consider a set of par bond yields, $\{y_n;\ y_n > 0, n = 1,\ldots,N\}$, where each y_n represents the par yield of a defaultable n-period fixed-rate bond. Assume the discrete-time recovery of face value. Let $y_0 \equiv 0$.*

(i) The general RF model will be consistent with the risky yield data iff $\forall n \in I$

$$\underline{y}_n^* \ \le\ y_n\ \le\ \overline{y}_n^* \tag{26}$$

where

$$\overline{y}_n^* = y_{n-1} + \frac{1 - w\,F(0,T_{n-1},T_n)}{h\ \Sigma_{n-1}} \tag{27}$$

$$\underline{y}_n^* = \frac{1 - F(0,T_{n-1},T_n)(1+h\,y_{n-1}) - (1-Q^{P_{n-1}}(0,T_{n-1}))(1-wF(0,T_{n-1},T_n))}{h\left(F(0,T_{n-1},T_n) + Q^{P_{n-1}}(0,T_{n-1})\Sigma_{n-1}\right)} + y_{n-1} \tag{28}$$

(ii) The constant-covariance RF model will be consistent with the risky yield data iff $\forall n \in I$

$$\underline{y}_n \ \le\ y_n\ \le\ \overline{y}_n \tag{29}$$

where

$$\overline{y}_n = \overline{y}_n^* \tag{30}$$

$$\underline{y}_n = y_{n-1} + \frac{1 - F(0,T_{n-1},T_n)(1+h\,y_{n-1})}{h\,(\Sigma_{n-1} + F(0,T_{n-1},T_n))} \tag{31}$$

Notice that unlike Theorem 1 which provides bounds on credit spreads, Theorem 2 establishes bounds of the par bond yields. (One can verify that $\forall n \in I$, $\overline{y}_n^* > \underline{y}_n^*$ and $\overline{y}_n > \underline{y}_n$.)

Notice also that the constant-covariance and the zero-covariance ($\rho_c = 0$) RF models produce the same risky yield bounds.

A.1.3 Implications for Model Specifications

As expected, it is more likely for the general RF model than the constant-covariance RF one to be consistent with a given set of risky par yields. This can be seen from the following:

$$
\begin{aligned}
\underline{y}_n - \underline{y}_n^* &= \frac{F(0,T_{n-1},T_n)(1-Q^{P_{n-1}}(0,T_{n-1}))[1-wF(0,T_{n-1},T_n)+(1-w)\Sigma_{n-1}+h\,y_{n-1}\Sigma_{n-1}]}{h\Big(\Sigma_{n-1}+F(0,T_{n-1},T_n)\Big)\Big(Q^{P_{n-1}}(0,T_{n-1})\Sigma_{n-1}+F(0,T_{n-1},T_n)\Big)} \\
&\geq 0 \qquad (32)
\end{aligned}
$$

where the inequality follows since $w \in [0,1]$ and $F(0,T_{n-1},T_n) \in (0,1) \forall n \in I$. The second equality in (32) realizes when $n = 1$. As expected, $\underline{y}_1^* = \underline{y}_1 = \ell(0,T_1)$.

Theorem 2 also imposes restrictions on the slopes of risky par yield curves with which the general or the constant-covariance RF model can be consistent. Let $f(0,T_{n-1},T_n)$ be the time-0 default-free discrete-compounding forward rate for $[T_{n-1},T_n]$. By definition,

$$
f(0,T_{n-1},T_n) = \frac{1}{h}\left(\frac{1}{F(0,T_{n-1},T_n)} - 1\right) \qquad (33)
$$

Corollary 3 *Let $\{y_n;\ y_n > 0, n = 1,\ldots,N\}$ be the set of par yields considered earlier in Theorem 2.*

(i) If $\forall n \in I$, $y_{n-1} \geq f(0,T_{n-1},T_n)$, then both the general and constant-covariance RF models

(a) will always be consistent with flat risky par yield curves;

(b) will be consistent with those upward-sloping risky par yield curves that satisfy the conditions: $y_n - y_{n-1} < \overline{y}_n^* - y_{n-1}, \forall n \in I$;

(c) will be consistent with those downward-sloping risky par yield curves that satisfy the conditions: $y_{n-1} - y_n < y_{n-1} - \underline{y}_n, \forall n \in I$.

(ii) If $\exists n \in \{2,\ldots,N\}$ s.t. $y_{n-1} < f(0,T_{n-1},T_n)$, then the constant-covariance RF model will not be consistent with flat or downward-sloping risky par yield curves.

Note that $y_{n-1} \geq f(0, T_{n-1}, T_n)\, \forall n \in \{2, \ldots, N\}$ - the if-conditions in Part (i) to hold - will occur if the default-free term structure is downward-sloping or mildly upward-sloping. In order for the condition in Part (ii) to hold, a default-free term structure with an extremely steep positive slope over $[T_{n-1}, T_n]$ would be necessary especially for a very high y_n from low-rated bonds. As a result, instances of that both default-free and risky par yield curves satisfy the conditions in Part (i) will be observed more often than instances of both satisfying the condition in Part (ii).

Now let's examine the "limiting" behavior of the risky par yield bounds. As shown in the proof of Corollary 2, $\Sigma_n \overset{n\uparrow\infty}{\longrightarrow} \infty$ and $Q^{P_n}(0, T_n)\Sigma_n \overset{n\uparrow\infty}{\longrightarrow} \infty$.[8] It then follows from (27), (28), (30) and (31) that the bounds on $|y_n - y_{n-1}|$, which equal $\overline{y}_n^* - y_{n-1}$ and $y_{n-1} - \underline{y}_n$ for upward- and for downward-sloping risky par yield curves respectively, all go to zero as n becomes infinity. As a result, the RF models of defaultable fixed-rate bond yields analyzed in this section can only be consistent with risky par yield curves that flatten out at the long end.

The behavior of risky yield spread curves generated from the RF models can be also analyzed if certain properties of the default-free term structure or the default-free par yield curve are known. For instance, suppose the default-free par yield curve is flat. In this case, $\rho_c = 0$ and we have a constant-covariance RF model. Let R_n denote the par yield of a default-free n-period coupon bond. It follows from (30) and (31) that the constant-covariance RF model is consistent with $\{y_n; n \in I\}$ if $\forall n \in I$

$$\frac{s_{n-1}^{\mathrm{RF}}\, \Sigma_{n-1}}{\Sigma_{n-1} + F(0, T_{n-1}, T_n)} \leq \; s_n^{\mathrm{RF}} \leq s_{n-1}^{\mathrm{RF}} + \frac{1 - w\, F(0, T_{n-1}, T_n)}{h\, \Sigma_{n-1}} \tag{34}$$

where $s_n^{\mathrm{RF}} = y_n - R_n$ represents the n-period par yield-spread and $s_0^{\mathrm{RF}} \equiv 0$. It is easy to see that spread curves, $\{s_n^{\mathrm{RF}}; n \in I\}$, satisfying the conditions given in (34) can be upward-sloping, downward-sloping or hump-shaped but have to be very flat at the long end.

The analysis with non-flat default-free par yield curves can be carried out in a similar fashion. For instance, if the default-free par yield curve is upward-sloping (downward-sloping), then, from the earlier analysis, both the general and constant-covariance RF models are consistent only when the observed credit spread curves are downward-sloping (upward-sloping) at the long end.

The explicit results of the par yield bounds obtained in Theorem 2 also allow for comparative

[8] These limiting results hold even though Σ_n analyzed in Corollary 2 is actually for floating-rate bounds.

statics. In particular, sensitivity analysis of these bounds to the recovery rate can be performed.

Proposition 3 *Consider the risky par yield boundaries, $\overline{y} := \{\overline{y}_n; n \in I\}$ and $\underline{y} := \{\underline{y}_n; n \in I\}$, given in (27) and (28), respectively. Then*

(i)

$$\frac{d\overline{y}_n}{dw} < 0, \qquad \forall n \in \{2, \ldots, N\}. \tag{35}$$

(ii)

$$\frac{d\underline{y}_n}{dw} \begin{cases} = 0 & n = 1, 2 \\ > 0 & \text{if } y_{n-1} > f(0, T_{n-1}, T_n),\ n = 3, \ldots, N \\ < 0 & \text{if } y_{n-1} < f(0, T_{n-1}, T_n),\ n = 3, \ldots, N \end{cases} \tag{36}$$

Proposition 3 shows that when the recovery rate increases, the upper boundary $\overline{y}$ always moves downward but the lower boundary $\underline{y}$ may move up or down depending on if $y_{n-1} > f(0, T_{n-1}, T_n)\, \forall n$. However, since as mentioned earlier, it would be much more likely to observe $y_{n-1} > f(0, T_{n-1}, T_n)\, \forall n$ than otherwise, the lower bound $\underline{y}_n$ normally shifts upward as the recovery rate increases. As a result, the constant-covariance RF model that assumes a zero recovery rate is more likely than other constant-covariance RF models to be consistent with the observed risky par yield curves. Sensitivity to w of the lower boundary $\underline{y}^*$ under the general RF model can be analyzed in a similar fashion but the analysis is much more involved and omitted here for brevity.

A.2 Models Based on Recovery of Market Value

In this subsection, for simplicity, we follow the reduced-form approach of Duffie and Singleton (1999) and Jarrow and Turnbull (1995) and assume that the default process follows a jump process. Under this approach, the recovery of market value convention leads to a simple formula for the value of a defaultable fixed-rate bond.

Let $V_B^{\mathrm{RM}}(0, T_n)$ denote the time-0 value of an n-period defaultable coupon bond. As shown in Duffie and Singleton (1999),

$$V_B^{\mathrm{RM}}(0, T_n) = E^{\mathbb{Q}}\left[\sum_{i=1}^{n} c_n h\ \Lambda(0, T_i) e^{-(1-w)\int_0^{T_i} \lambda_t dt} + \Lambda(0, T_n) e^{-(1-w)\int_0^{T_n} \lambda_t dt}\right] \tag{37}$$

where λ is the intensity process and the remaining variables and parameters are defined as before. Under the forward measure, equation (37) can be rewritten as follows:

$$V_B^{\mathrm{RM}}(0,T_n) = c_n h \sum_{i=1}^{n} P(0,T_i)\overline{Q}^{P_i}(0,T_i) + P(0,T_n)\overline{Q}^{P_n}(0,T_n) \tag{38}$$

where

$$\overline{Q}^{P_i}(0,T_i) := E^{\mathbb{P}_i}\left[e^{-(1-w)\int_0^{T_i}\lambda_t dt}\right] \tag{39}$$

Unless $w = 0$, $\overline{Q}^{P_i}(0,T_i)$ does not represent a survival probability (under the T_i-forward measure). Notice that the coefficients of $\overline{Q}^{\cdot}(0,\cdot)$ in (38) are the same as those of $Q(0,\cdot)$ for $w = 0$ in (24). This implies that risky par yield data alone cannot distinguish a model that assumes zero recovery of face value from a model that assumes non-zero recovery of market value. (Using a parametric model, Duffie and Singleton (1999) also demonstrate that there is little difference in calculating par-bond spreads between the recovery of face value at default time and recovery of market value conventions.)

Applying (38) to par bonds leads to

$$(1 + y_n h)\ \frac{\overline{Q}^{P_n}(0,T_n)}{\overline{Q}^{P_{n-1}}(0,T_{n-1})} = \frac{1 + h\ (y_{n-1} - y_n)\overline{\Sigma}_{n-1}}{F(0,T_{n-1},T_n)}, \qquad n = 1,\dots,N, \tag{40}$$

where

$$\overline{\Sigma}_n := \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\overline{Q}^{P_i}(0,T_i)}{\overline{Q}^{P_n}(0,T_n)} \frac{P(0,T_i)}{P(0,T_n)} & n \geq 1 \\ 0 & n = 0 \end{cases} \tag{41}$$

Since $\forall i \in I$, $\overline{Q}^{P_i}(0,T_i) \in [0,1]$, equation (40) can be used to establish bounds on risky par yields even though $\overline{Q}^{P_i}(0,T_i)$ with $w > 0$ does not represent a survival probability. A tighter upper bound on $\overline{Q}^{\cdot}(0,\cdot)$ and, as a result, a tighter lower bound on risky yields, $\{y_i; i \in I\}$, can be obtained if the following condition holds

$$cov^{\mathbb{Q}}\left(\frac{d\mathbb{P}_i}{d\mathbb{Q}}, e^{-(1-w)\int_0^{T_i}\lambda_t dt}\right) = constant, \qquad \forall i \in I \tag{42}$$

Under this condition, $\overline{Q}^{P_i}(0,T_i) - \overline{Q}^{P_{i-1}}(0,T_{i-1}) = Q^{P_i}(0,T_i) - Q^{P_{i-1}}(0,T_{i-1})$ and it follows that $\overline{Q}^{P_i}(0,T_i) \leq \overline{Q}^{P_{i-1}}0,T_{i-1})$ since $Q^{P_i}(0,T_i) \leq Q^{P_{i-1}}(0,T_{i-1})\,\forall i \in \{2,\dots,N\}$. Given the bounds of $\overline{Q}^{\cdot}(0,\cdot)$, we can examine if a model with recovery of market value is consistent with the observed risky par yields. This analysis can be handled similarly to the case of recovery of face value in Section A.1 and is omitted here for brevity.

B Proofs

B.1 Proof of Proposition 1

From (5), we have

$$
\begin{aligned}
V(0,T_n) &= \sum_{i=1}^{n} P(0,T_i)\left[Q(0,T_i)(F(0,T_{i-1},T_i)^{-1}-1)+h\,\gamma_{i-1,i}\right] \\
&\quad + s_n\, h \sum_{i=1}^{n} P(0,T_i)Q(0,T_i) + P(0,T_n)Q(0,T_n)
\end{aligned} \tag{43}
$$

and

$$
\begin{aligned}
V(0,T_{n-1}) &= \sum_{i=1}^{n-1} P(0,T_i)\left[Q(0,T_i)(F(0,T_{i-1},T_i)^{-1}-1)+h\,\gamma_{i-1,i}\right] \\
&\quad + s_{n-1}\, h \sum_{i=1}^{n-1} P(0,T_i)Q(0,T_i) + P(0,T_{n-1})Q(0,T_{n-1})
\end{aligned} \tag{44}
$$

where $\gamma_{i-1,i}$ is as defined before. Since $V(0,T_n)=1=V(0,T_{n-1})$ for par bonds, subtracting (44) from (43) and rearranging terms yields

$$
\begin{aligned}
& s_n\, h\, P(0,T_n)Q(0,T_n) + P(0,T_n)\left(\frac{Q(0,T_n)}{F(0,T_{n-1},T_n)} + h\,\gamma_{n-1,n}\right) = \\
&\qquad P(0,T_{n-1})Q(0,T_{n-1}) - h\,(s_n - s_{n-1})\sum_{i=1}^{n-1} P(0,T_i)Q(0,T_i)
\end{aligned} \tag{45}
$$

If $Q(0,T_{n-1})>0$, then (45) can be rewritten as follows.

$$
\begin{aligned}
& \frac{Q(0,T_n)}{Q(0,T_{n-1})}\left[1+h\,s_n\,F(0,T_{n-1},T_n)\right] = \\
&\qquad 1 - h\,\gamma_{n-1,n}\frac{F(0,T_{n-1},T_n)}{Q(0,T_{n-1})} - h\,(s_n-s_{n-1})\sum_{i=1}^{n-1}\frac{P(0,T_i)}{P(0,T_{n-1})}\frac{Q(0,T_i)}{Q(0,T_{n-1})}
\end{aligned} \tag{46}
$$

Equation (10) in Proposition 1 then follows. This completes the proof. □

B.2 Proof of Lemma 1

By definition,

$$Q(0,T_n) = Q^{\mathrm{RN}}(0,T_n) + cov\left(\frac{d\mathbb{F}_n}{d\mathbb{Q}}, I_{\{T_n<\tau\}}\right), \qquad n = 1,\ldots,N \tag{47}$$

Then it follows from (1) and Assumption 1 that

$$Q(0,T_n) - Q(0,T_{n-1}) = Q^{\mathrm{RN}}(0,T_n) - Q^{\mathrm{RN}}(0,T_{n-1}), \quad n = 2,\ldots,N \tag{48}$$

Since $\forall n \in \{2,\ldots,N\}$, $Q^{\mathrm{RN}}(0,T_n) \leq Q^{\mathrm{RN}}(0,T_{n-1})$, Eq. (48) implies that

$$Q(0,T_n) \leq Q(0,T_{n-1}), \quad n = 2,\ldots,N \tag{49}$$

□

B.3 Proof of Theorem 1

(i) A given credit risk model will not be consistent with $\{s_n; n \in I\}$ if the model-implied default/survival probabilities are not well defined. Namely, for $Q(0,T_n)\,\forall n \in I$ to be well defined, it should satisfy conditions: $0 \leq Q(0,T_n) \leq 1$. Applying these two conditions to (10) yields (14) and (15).

(ii) Under Assumption 1, it follows from Lemma 1 that (13) holds. Applying (13) to (10) yields the lower bound given in (16). □

B.4 Proof of Corollary 1

(i) If $\exists n \in I$ s.t. $\gamma_{n-1,n} > \gamma^u_{n-1,n}$, one can see from (15) that $\overline{s}^*_n < s_{n-1}$. It then follows that $s_n > \overline{s}^*_n$ since $s_n \geq s_{n-1}$ with an upward-sloping spread curve. By Theorem 1, the model specification in this case is inconsistent with the spread data.

(ii) If $\exists n \in I$ s.t. $\gamma_{n-1,n} < \gamma^d_{n-1,n}$, one can see from (14) that $\underline{s}^*_n > s_{n-1}$. It then follows that $s_n < \underline{s}^*_n$ since $s_n \leq s_{n-1}$ with a downward-sloping spread curve. By Theorem 1, the model specification in this case is inconsistent with the spread data.

(iii) Under the assumptions made here, either $s_n = s_{n-1} > \overline{s}_n^*$ or $s_n = s_{n-1} < \underline{s}_n^*$. The claim then follows directly from (i) and (ii). □

B.5 Proof of Corollary 2

By definition

$$\Sigma_n = \sum_{i=1}^{n} \frac{Q(0,T_i)}{Q(0,T_n)} \frac{P(0,T_i)}{P(0,T_n)} \geq \frac{Q(0,T_1)}{Q(0,T_n)} \frac{P(0,T_1)}{P(0,T_n)}, \qquad n = 1,\ldots,N \tag{50}$$

Since both $Q(0,T_n)$ and $P(0,T_n)$ go to zero as n goes to infinity, it follows from (50) that Σ_n is large for high values of n. It is easy to see from (50) that $Q(0,T_n)\Sigma_n$ is also large for high values of n. Given these observations, the claims in Corollary 2 follow directly from (14)-(17). □

B.6 Lemma 2 and Its Proof

Proof of Proposition 2 uses the following lemma.

Lemma 2 *Consider a set of credit spreads from defaultable par floating-rate bonds:* $\{s_n; s_n \geq 0, n = 1,\ldots,N\}$. *Suppose* $s_i < \min(\overline{s}_i^0, \overline{s}_i), i = 2,\ldots,n-1$. *We have*

$$\Sigma_n - \Sigma_n^0 = \Sigma_{n-1}^0 \frac{P(0,T_{n-1})}{P(0,T_n)} \frac{Q^0(0,T_{n-1})}{Q^0(0,T_n)} \frac{\overline{s}_n^0 - \overline{s}_n}{\overline{s}_n - s_n} \tag{51}$$

where

$$Q^0(0,T_n) = Q(0,T_n) \mid_{\gamma_{j-1,j}=0\,\forall j\in\{1,\ldots,n\}}, \qquad n = 1,\ldots,N \tag{52}$$

Proof. Using (52), we can obtain the following alternative expression of Σ_n^0:

$$\Sigma_n^0 = \sum_{i=1}^{n} \frac{Q^0(0,T_i)}{Q^0(0,T_n)} \frac{P(0,T_i)}{P(0,T_n)}, \qquad n = 1,\ldots,N \tag{53}$$

It follows from (11) and (53) that

$$\Sigma_n^0 = 1 + \Sigma_{n-1}^0 \frac{Q^0(0,T_{n-1})}{Q^0(0,T_n)} \frac{P(0,T_{n-1})}{P(0,T_n)} \tag{54}$$

$$\Sigma_n = 1 + \Sigma_{n-1} \frac{Q(0,T_{n-1})}{Q(0,T_n)} \frac{P(0,T_{n-1})}{P(0,T_n)} \tag{55}$$

Subtracting Σ_n^0 from Σ_n yields

$$\Sigma_n - \Sigma_n^0 = \frac{P(0,T_{n-1})}{P(0,T_n)} \left(\Sigma_{n-1} \frac{Q(0,T_{n-1})}{Q(0,T_n)} - \Sigma_{n-1}^0 \frac{Q^0(0,T_{n-1})}{Q^0(0,T_n)} \right) \tag{56}$$

Using (17), we can rewrite (46) as follows

$$\frac{Q(0,T_n)}{Q(0,T_{n-1})}[1 + h\, s_n\, F(0,T_{n-1},T_n)] = h\, \Sigma_{n-1}\, (\overline{s}_n - s_n) \tag{57}$$

Similarly, we have

$$\frac{Q^0(0,T_n)}{Q^0(0,T_{n-1})}[1 + h\, s_n\, F(0,T_{n-1},T_n)] = h\, \Sigma_{n-1}^0\, (\overline{s}_n^0 - s_n) \tag{58}$$

Dividing (57) by (58) yields

$$\left(\frac{Q(0,T_n)}{Q(0,T_{n-1})} \right) \Big/ \left(\frac{Q^0(0,T_n)}{Q^0(0,T_{n-1})} \right) = \frac{\Sigma_{n-1}}{\Sigma_{n-1}^0} \frac{\overline{s}_n - s_n}{\overline{s}_n^0 - s_n} \tag{59}$$

Substituting (59) into (56) and simplifying yields (51). □

B.7 Proof of Proposition 2

Here we prove a slightly more general version of Part (i) of Proposition 2 by allowing $\gamma_{n-1,n}$'s to be zero. Part (ii) of the proposition can be shown in a similar fashion and its proof is omitted here for brevity.

We prove Part (i) by induction. First, we show that the claim holds for $n = 2$, i.e. if $\gamma_{1,2} \le 0$, then $\overline{s}_2 \ge \overline{s}_2^0$. Subtracting (22) from (17) yields

$$\overline{s}_n - \overline{s}_n^0 = \frac{1}{h} \left(\frac{1}{\Sigma_{n-1}} - \frac{1}{\Sigma_{n-1}^0} \right) - \gamma_{n-1,n} \frac{1}{\Sigma_{n-1}} \frac{F(0,T_{n-1},T_n)}{Q(0,T_{n-1})} \tag{60}$$

Setting $n = 2$ in (60) and using the fact that $\Sigma_1 = \Sigma_1^0 = 1$, we have

$$\overline{s}_2 - \overline{s}_2^0 = -\gamma_{1,2} \frac{1}{\Sigma_1} \frac{F(0,T_1,T_2)}{Q(0,T_1)} \ge 0 \tag{61}$$

Next, we show that the proposition holds for $n = 3$, i.e. if $\gamma_{i-1,i} \le 0, i = 2, 3$, then $\overline{s}_3 \ge \overline{s}_3^0$. Given $\gamma_{1,2} \le 0$, it follows from (51) in Lemma 2 and (61) that

$$\Sigma_2 - \Sigma_2^0 \leq 0 \tag{62}$$

Given (62) and $\gamma_{2,3} \leq 0$ (by assumption), we have from (60)

$$\overline{s}_3 - \overline{s}_3^0 = \frac{1}{h}\left(\frac{1}{\Sigma_2} - \frac{1}{\Sigma_2^0}\right) - \gamma_{2,3}\,\frac{1}{\Sigma_2}\,\frac{F(0,T_2,T_3)}{Q(0,T_2)} \geq 0 \tag{63}$$

In a similar fashion, we can show that if the proposition holds for $n-1$, then it also does for n. This completes the proof. □

B.8 Proof of Theorem 2

We shall give the proof of the "⇐" part only. The proof of the "⇒" part is straightforward.

(i) Inequalities in (26) follow directly from the restrictions that $0 \leq Q(0,T_n) \leq 1\,\forall n \in I$ and (25). To verify $\overline{y}_n^* > \underline{y}_n^*\ \forall n \in I$, we have from (27) and (28)

$$\overline{y}_n^* - \underline{y}_n^* = \frac{1}{h}\,\frac{F(0,T_{n-1},T_n)\left(1 - w + h\ y_{n-1} + \frac{1-w\ F(0,T_{n-1},T_n)}{\Sigma_{n-1}}\right)}{Q(0,T_{n-1})\Sigma_{n-1} + F(0,T_{n-1},T_n)}$$

Since $0 \leq w \leq 1$ and $0 < F(0,T_{n-1},T_n) < 1$, $\overline{y}_n^* > \underline{y}_n^*$.

(ii) Under Assumption 1, we have $0 \leq Q(0,T_n) \leq Q(0,T_{n-1})\,\forall n \in I$. Inequalities in (29) then follow directly from (25). To show $\overline{y}_n > \underline{y}_n\ \forall n \in I$, we have from (30) and (31)

$$\begin{aligned}\overline{y}_n - \underline{y}_n &= \frac{1}{h}\,\frac{1}{\Sigma_{n-1} + F(0,T_{n-1},T_n)}\left(1 - w + h\ y_{n-1} + \frac{1-w\ F(0,T_{n-1},T_n)}{\Sigma_{n-1}}\right)\\ &> 0\end{aligned}$$

where the inequality follows since $0 \leq w \leq 1$ and $F(0,T_{n-1},T_n) < 1$. □

B.9 Proof of Corollary 3

(i) Using (27), (28), (31) and (33), we have

$$\overline{y}_n^* - y_n = y_{n-1} - y_n + \frac{1 - w\,F(0,T_{n-1},T_n)}{h\,\Sigma_{n-1}} \tag{64}$$

$$y_n - \underline{y}_n^* = y_n - y_{n-1} + \frac{F(0,T_{n-1},T_n)}{F(0,T_{n-1},T_n) + Q(0,T_{n-1})\,\Sigma_{n-1}}$$
$$\cdot \left[(y_{n-1} - f(0,T_{n-1},T_n)) + (1 - Q(0,T_{n-1})) \frac{1 - w\,F(0,T_{n-1},T_n)}{h\,F(0,T_{n-1},T_n)} \right] \quad (65)$$
$$y_n - \underline{y}_n = y_n - y_{n-1} + \frac{(y_{n-1} - f(0,T_{n-1},T_n))\,F(0,T_{n-1},T_n)}{\Sigma_{n-1} + F(0,T_{n-1},T_n)} \quad (66)$$

By Theorem 2, it suffices to show that in every case, the conditions given in (26) and (29) are satisfied.

(a) Given $y_{n-1} \geq f(0,T_{n-1},T_n)$ and $y_{n-1} = y_n \,\forall n \in \{2,\ldots,N\}$, it is easy to see from (64) and (65) that $\forall n \in I, \overline{y}_n^* > y_n$ and $y_n > \underline{y}_n^*$. Namely, $\underline{y}_n^* \leq y_n \leq \overline{y}_n^* \,\forall n \in I$. Similarly, one can show that $\underline{y}_n \leq y_n \leq \overline{y}_n \,\forall n \in I$ using (66) and the fact that $\overline{y}_n = \overline{y}_n^* \forall n \in I$.

(b) For upward-sloping risky yield curves, $y_{n-1} \leq y_n \,\forall n \in I$. One can see from (65) and (66) that $y_n > \underline{y}_n^*$ and $y_n > \underline{y}_n$. Given $y_n - y_{n-1} < \overline{y}_n^* - y_{n-1}, \forall n \in I$, it is obvious that $y_n < \overline{y}_n^* \,\forall n \in I$.

(c) For downward-sloping risky yield curves, $y_{n-1} \geq y_n \,\forall n \in \{2,\ldots,N\}$. One can see from (64) that $y_n < \overline{y}_n^* \,\forall n$. Since $y_{n-1} - y_n < y_{n-1} - \underline{y}_n \,\forall n \in \{2,\ldots,N\}$ by assumption, $y_n > \underline{y}_n \,\forall n$. It follows from (32) that $y_n > \underline{y}_n^* \,\forall n$.

(ii) Given $\exists n \in I$ s.t. $y_{n-1} < f(0,T_{n-1},T_n)$ and $\forall n \in \{2,\ldots,N\}, y_{n-1} \geq y_n$, it follows from (66) that $y_n < \underline{y}_n$. This violates one of the conditions given in (29). By Theorem 2, the constant-covariance RF model is inconsistent in this case. □

B.10 Proof of Proposition 3

(i) Differentiating w.r.t. w on both sides of (30) yields

$$\frac{d\overline{y}_{n+1}}{dw} = -\frac{F(0,T_n,T_{n+1})}{h\,\Sigma_n} \left(1 + (1 - w\,F(0,T_n,T_{n+1})) \frac{d \ln \Sigma_n}{dw} \right), \quad n = 1,\ldots,N-1. \quad (67)$$

To calculate $d\Sigma_n/dw$, we first use (30) and (55) to rewrite (25) as follows:

$$\frac{1}{\Sigma_n - 1} = \frac{h\,(\overline{y}_n - y_n)}{1 + h\,y_n - w}, \qquad n = 1,\ldots,N. \quad (68)$$

Differentiating w.r.t. w on both sides of (68) then and rearranging leads to

$$\frac{d\Sigma_n}{dw} = -\frac{\Sigma_n - 1}{1 + h\,y_n - w}\left(1 + \frac{1 + h\,y_n - w}{\overline{y}_n - y_n}\,\frac{d\overline{y}_n}{dw}\right) \tag{69}$$

Substituting (67) with $n = n-1$ into the RHS of (69) and then using (25) arrives

$$\frac{d\Sigma_n}{dw} = -\frac{\Sigma_n - 1}{1 + h\,y_n - w}\left[1 - \frac{Q(0,T_n)}{Q(0,T_{n-1})}\left(1 + \frac{1 - w\,F(0,T_{n-1},T_n)}{F(0,T_{n-1},T_n)}\,\frac{d\ln\Sigma_{n-1}}{dw}\right)\right] \tag{70}$$

It is easy to see from (70) that $d\Sigma_2/dw > 0$ since $\Sigma_1 = 1$ and $Q(0,T_n) \leq Q(0,T_{n-1})\,\forall n \in I$ (c.f. Lemma 1). Suppose $d\Sigma_{n-1}/dw \geq 0, n \geq 3$. It is then obvious from (70) that $d\Sigma_n/dw \geq 0$ given that $w\,F(0,T_n,T_{n+1}) \in [0,1)$ and $Q(0,T_n) \leq Q(0,T_{n-1})$. By induction, $d\Sigma_n/dw \geq 0\,\forall n \in I$. It then follows from (67) that $d\overline{y}_n/dw < 0\,\forall n \in \{2,\ldots,N\}$.

(ii) Differentiating w.r.t. w on both sides of (31) yields

$$\begin{aligned}\frac{d\underline{y}_n}{dw} &= -\frac{d\Sigma_{n-1}}{dw}\,\frac{1 - F(0,T_{n-1},T_n)(1 + h\,y_{n-1})}{h\left(\Sigma_{n-1} + F(0,T_{n-1},T_n)\right)^2} \\ &= -\frac{d\Sigma_{n-1}}{dw}\,\frac{(f(0,T_{n-1},T_n) - y_{n-1})\,F(0,T_{n-1},T_n)}{\left(\Sigma_{n-1} + F(0,T_{n-1},T_n)\right)^2}, \qquad \forall n \in I\end{aligned} \tag{71}$$

where the last equality is obtained using (33) and $n \in I$. Since $d\Sigma_0/dw = 0 = d\Sigma_1/dw$, it is obvious from (71) that $d\underline{y}_n/dw = 0, n = 1,2$. $\forall n \in \{3,\ldots,N\}, d\Sigma_{n-1}/dw > 0$ (see the proof of part (i)), and it follows from (71) that $d\underline{y}_n/dw$ and $y_{n-1} - f(0,T_{n-1},T_n)$ have the same sign. This completes the proof. □

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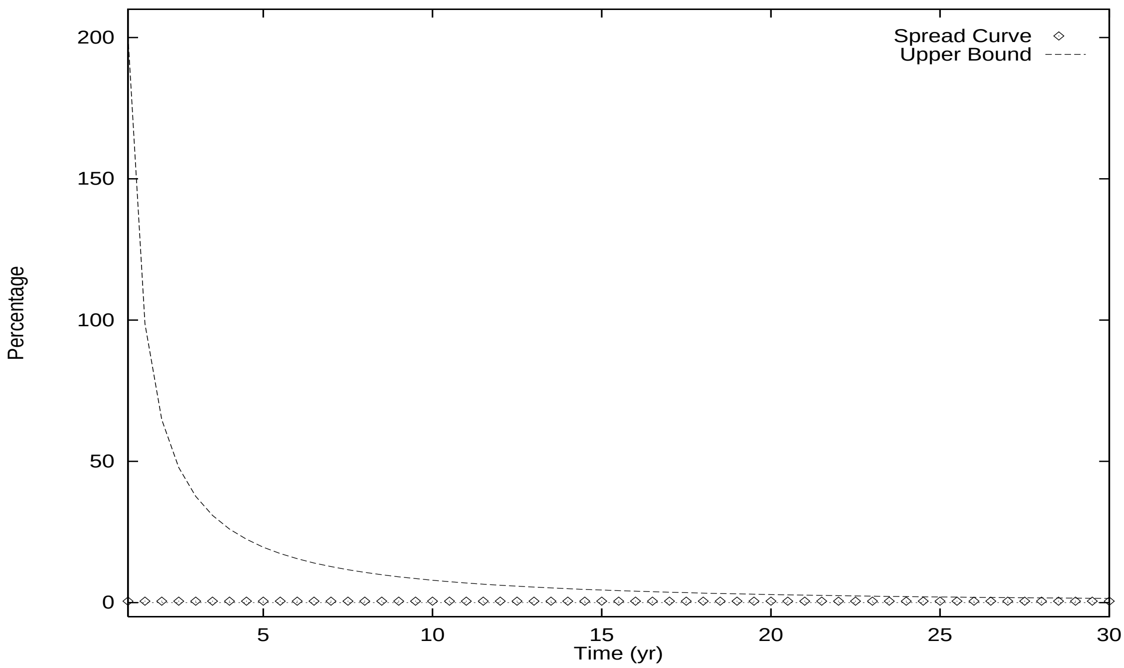
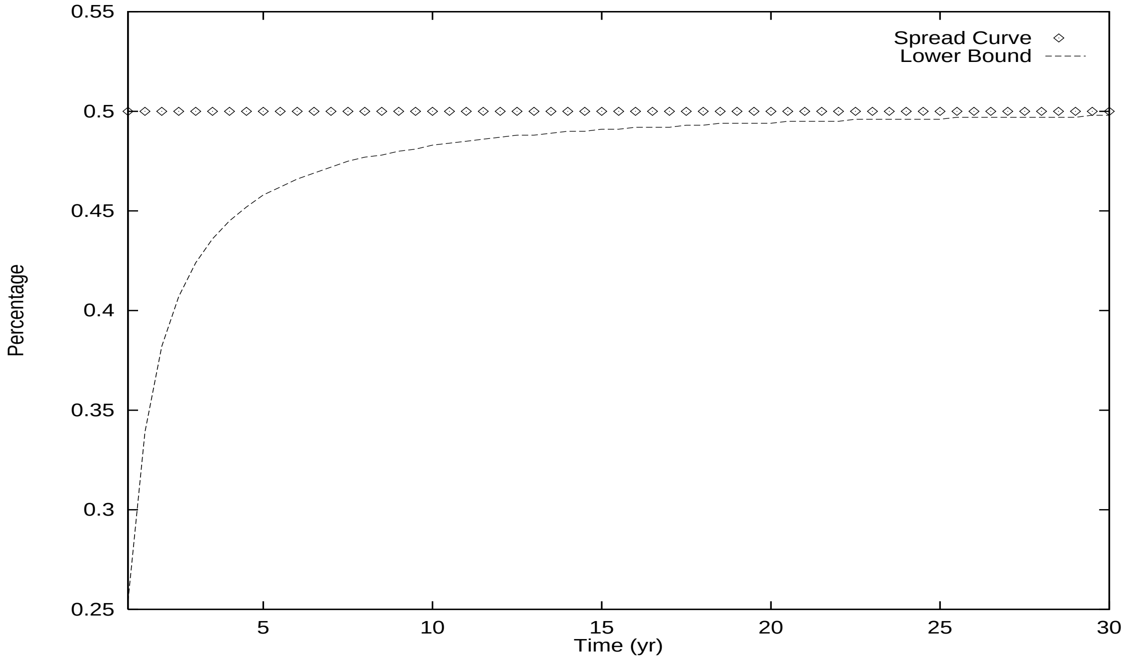
Table 1: Credit spread curves of SBP and Softbank

	07/26/99		06/08/00	
Term (month)	Treasury CMT yield (%)	Spread of SBP (bp)	Treasury CMT yield (%)	Spread of Softbank (bp)
3	4.88	25	5.59	35.78
6	5.03	25	5.71	52.35
12	5.29	25	5.82	68.91
24	5.61	20	5.94	90.48
36	5.70	15	6.00	100.17
48	5.89	10	6.05	107.04
60	5.85	5	6.09	122.38
84	6.01	0	6.14	140.42
120	5.89	0	6.20	168.94
240	6.36	0	6.31	335.51
360	6.18	13	6.38	395.20

Table 1 reports the data used to calibrate credit spread curves of two defaultable floating-rate bond issues, one from Santander BanCorp (SBP) and the other from Softbank. The spreads reported here are measured relative to the US constant maturity Treasury (CMT) rates. The spreads of SBP were updated on July 26, 1999 and those of Softbank on June 8, 2000.



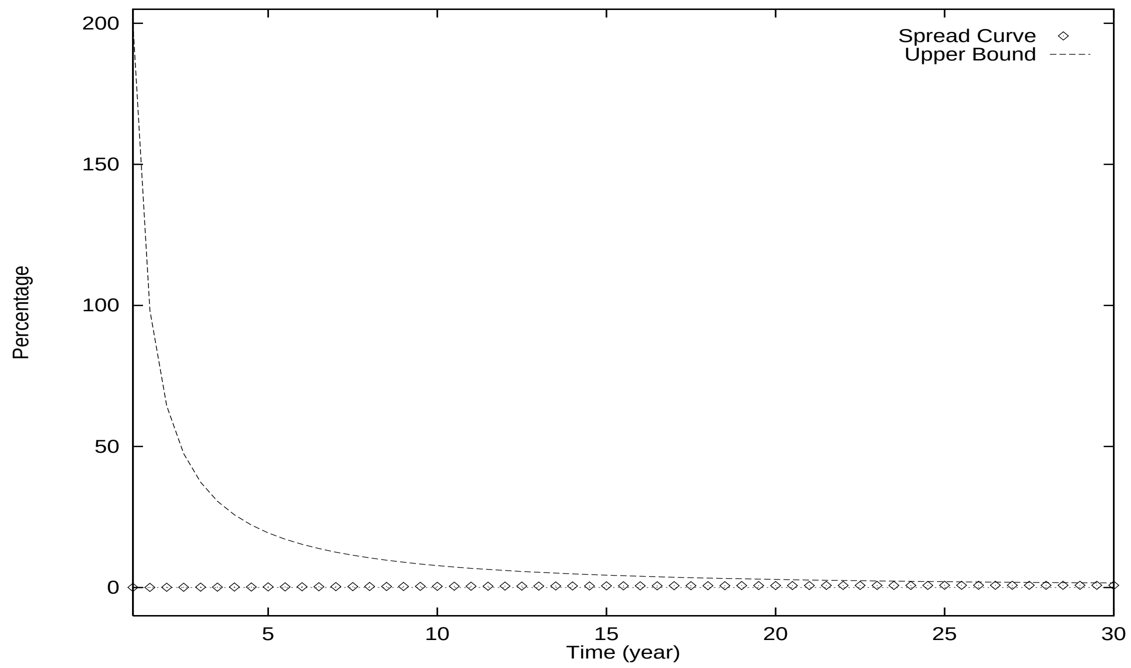
(a) The Credit Spread Curve and Its Upper Bound



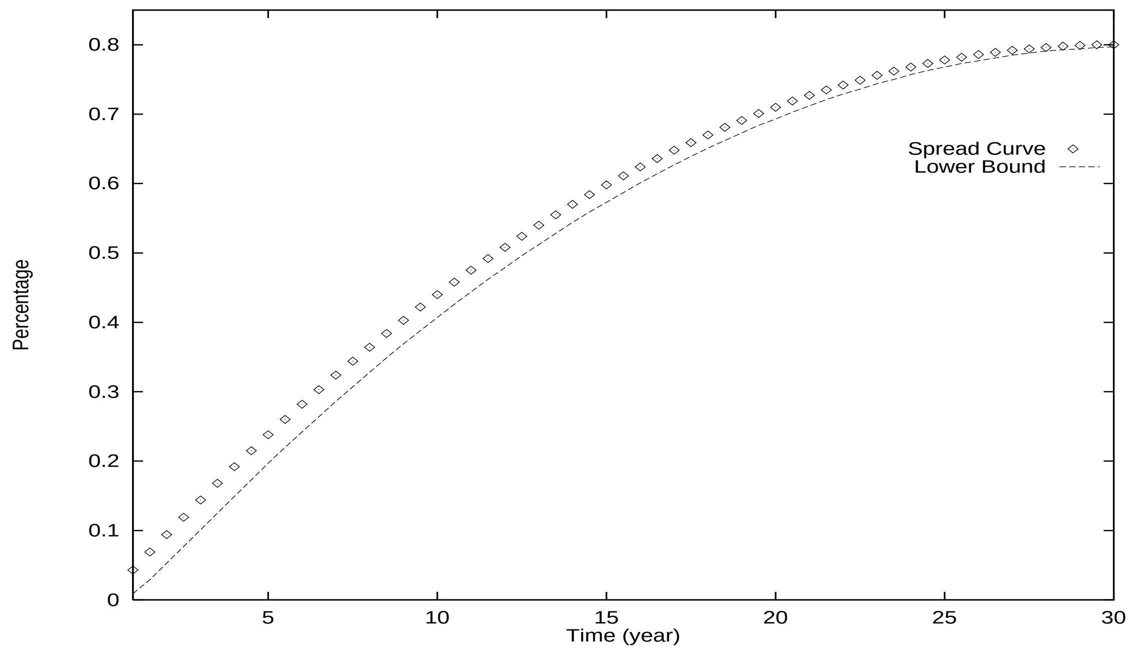
(b) The Credit Spread Curve and Its Lower Bound

Figure 1: A Flat Credit Spread Curve and Its Bounds from the Zero-Covariance Model

The spread curve is from defaultable par floating-rate bonds. The credit spread boundaries are generated using the zero-covariance model ($\rho_c = 0$). The model is correctly specified since the spread curve lies between its two bounds.



(a) The Credit Spread Curve and Its Upper Bound



(b) The Credit Spread Curve and Its Lower Bound

Figure 2: An Upward-Sloping Credit Spread Curve and Its Bounds from the Zero-Covariance Model

The spread curve is from defaultable par floating-rate bonds. The credit spread boundaries are generated using the zero-covariance model ($\rho_c = 0$). The model is correctly specified since the spread curve lies between its two bounds.

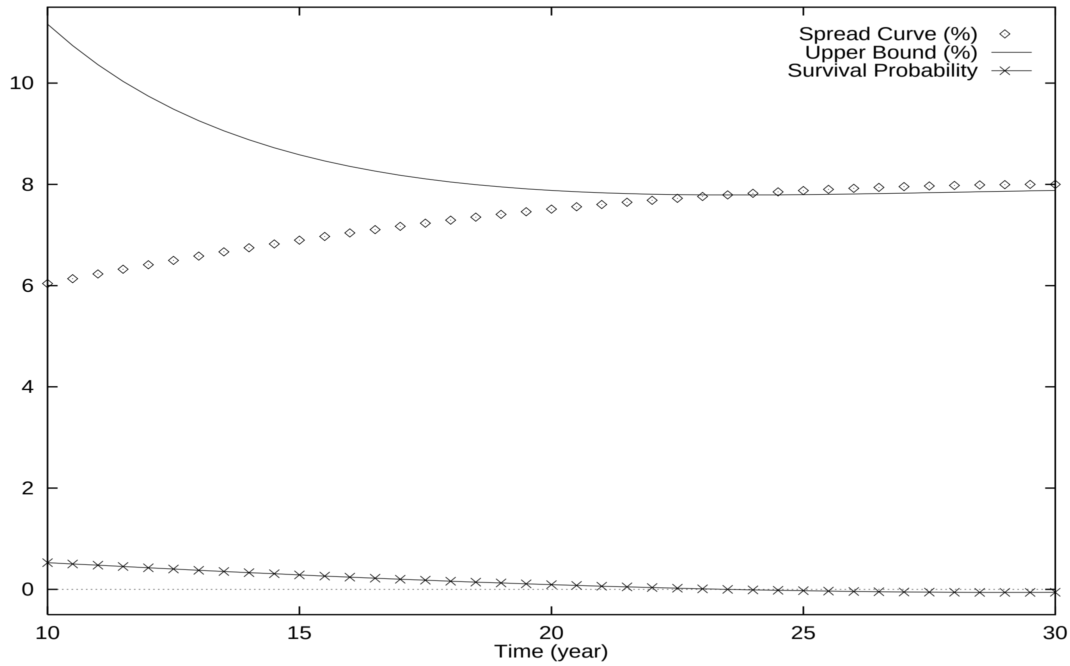
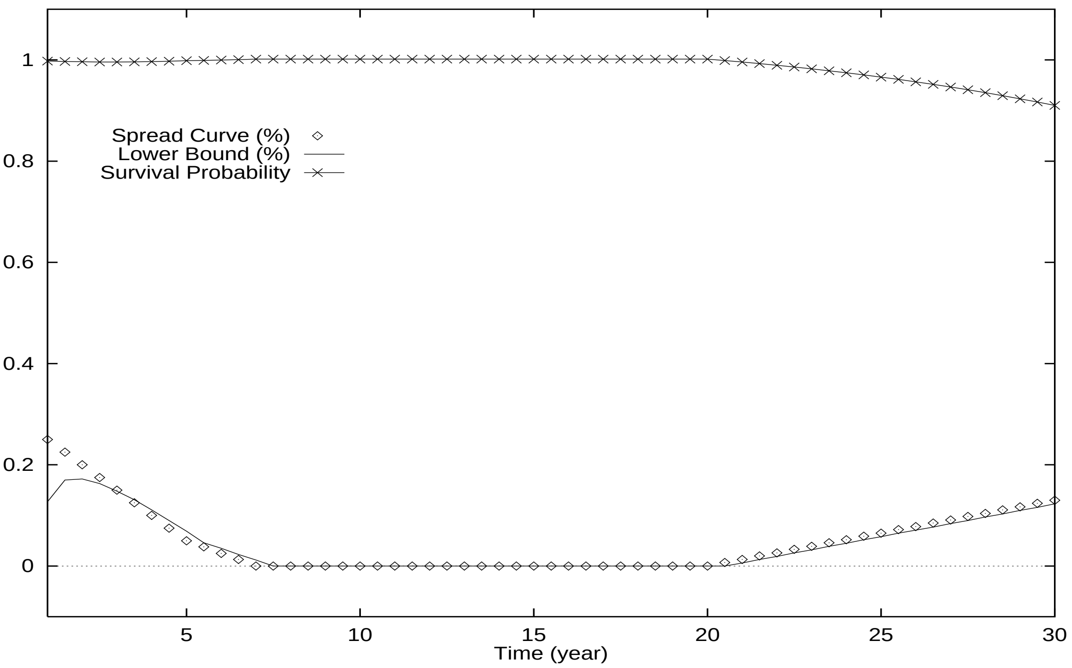
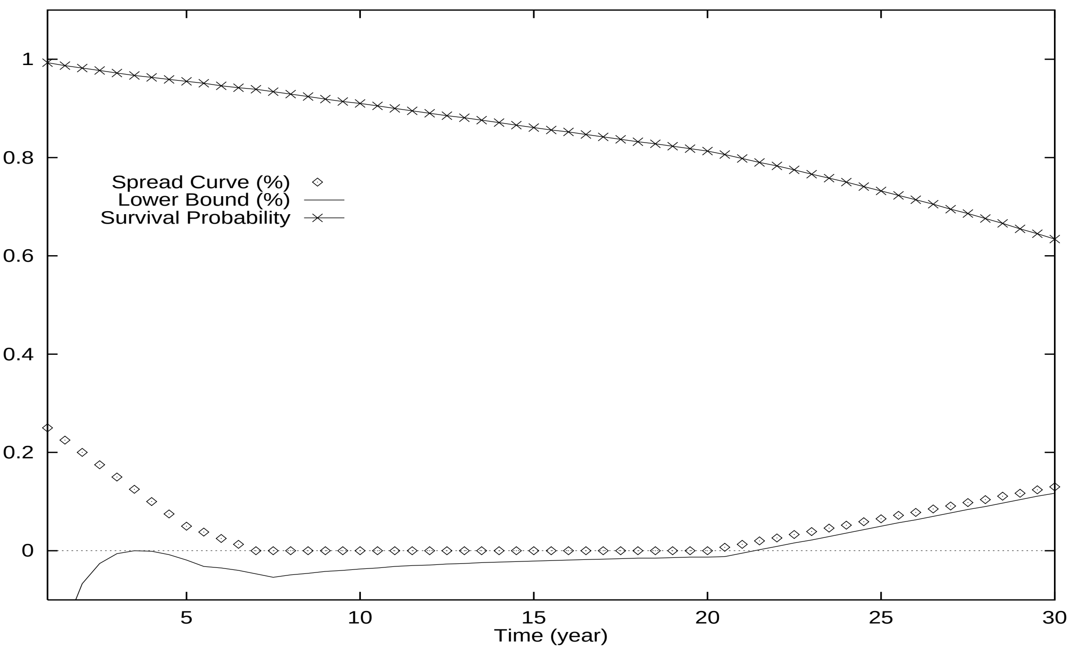


Figure 3: A Steeply Upward-Sloping Credit Spread Curve, and Its Upper Bound and Implied Survival Probability Curve from the Zero-Covariance Model

The spread curve is from defaultable par floating-rate bonds. The credit spread boundaries and the implied survival probability curve are generated using the zero-covariance model ($\rho_c = 0$). The model is misspecified since the credit spread curve crosses its upper bound around year 23.

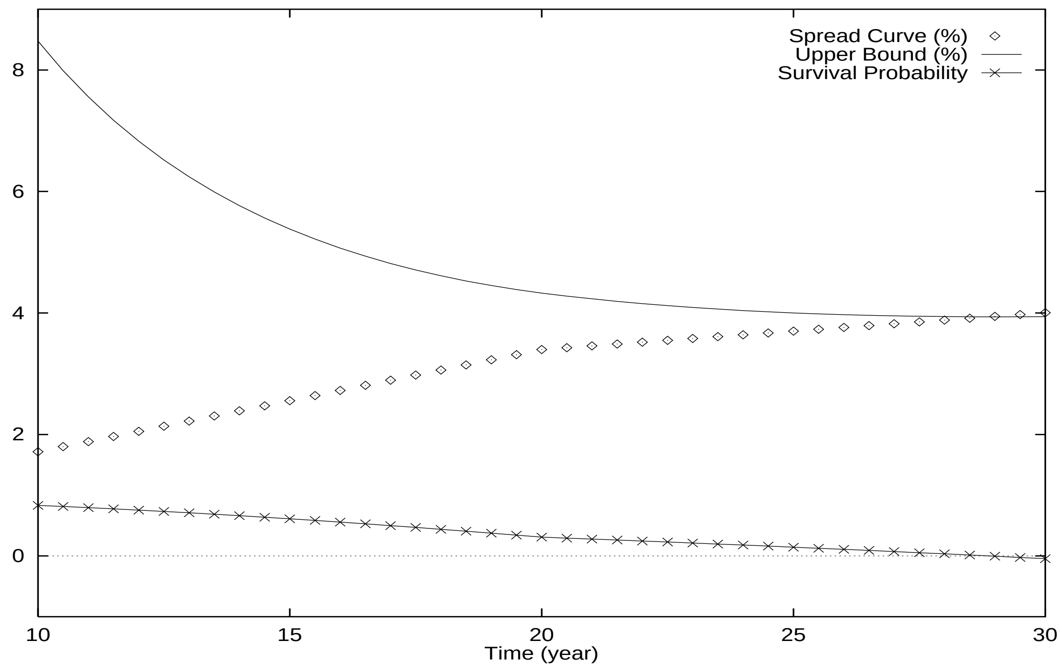


(a) The zero-covariance model ($\rho_c = 0$)

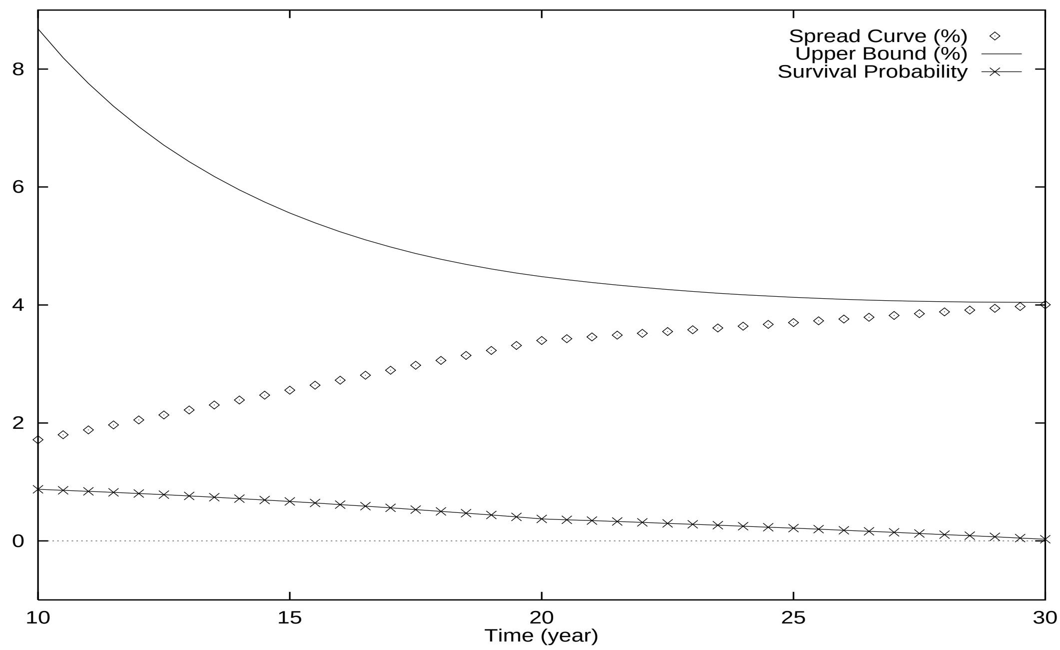


(b) The constant-gamma model with $\gamma_0 = 0.01$

Figure 4: The SBP Credit Spread Curve, and Its Lower Bounds and Survival Probability Curves
The Santander BanCorp (SBP) credit spread curve is generated using data on July 26, 1999 (reported in Table 1). Figures 4(a) and (b) plot the lower bound and the survival probability curve generated using the zero-covariance model and constant-gamma model with $\gamma_0 = 0.01$, respectively.



(a) The zero-covariance model ($\rho_c = 0$)



(b) The constant-gamma model with $\gamma_0 = -0.005$

Figure 5: The Softbank Credit Spread Curve, and Its Upper Bounds and Survival Probability Curves
The Softbank credit spread curve is generated using data on June 8, 2000 (reported in Table 1). Figures 5(a) and (b) plot the upper bound and the implied survival probability curve generated using the zero-covariance model and the constant-gamma model with $\gamma_0 = -0.005$, respectively.